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**YUFEI ZHAO:** Oftentimes, in applying the probabilistic method, we'll need to understand whether a certain random structure typically has a property that we're looking for. In this video, we'll look at one such property and understand how to analyze whether a certain random graph has the property of containing a triangle with high probability.

The objects that we're going to be looking at today is the Erdos-Rényi random graph, commonly referred to as G(n,p). And this is the graph consisting of placing n vertices. And for each pair of vertices, we put down an edge between them with probability p independently for all pairs of vertices. In other words, we flip a biased coin that comes up heads with probability p independently for each possible edge and put down the edge randomly as such.

This is a random graph. It's a random structure. It's a random object. And we would like to understand various properties of this random graph. And the property that we're going to look at in this video is the following. Does this random graph typically contain a triangle?

In the rest of this video, we'll be looking at p as a function of n. So p is allowed to change, depending on n. Well, we can denote this dependence by a subscript, but out of convenience, we'll drop the subscript for the rest of the video.

So G(n,p) Does it typically contain a triangle? Well, by "typically" I mean the following mathematically precise meaning. "Typically" means with probability that approaches 1 as n goes to infinity. So the statements are going to be looking at are asymptotic as n becomes large.

The main theorem that we'll prove in this video is the following. Well, there are two statements. I have conveniently labeled them theorem 0 and theorem 1. Theorem 0, sometimes referred to as the 0 statement, is the statement about not having any triangles, having zero triangles.

And it says that if n times p-- remember, p is a function of n. p is allowed to depend on n. If n times p approaches 0 as n goes to infinity, then this random graph G(n,p) is triangle-free with probability approaching 1 as n goes to infinity. In other words, with high probability, this random graph does not have any triangles.

This makes sense in that this graph is pretty sparse when p is small. So this statement basically says that p is much less than 1 over n, in terms of the growth rate. So when p is very small, you expect not so many edges. And it turns out that there's not going to be a triangle typically in this random graph.

The second part is 0 and 1. The 1 statement. And it says that when (n,p) approaches infinity, again, p is a function of n, then this random graph G(n,p) typically contains a triangle. IE, it contains a triangle with probability that approaches 1 as n goes to infinity. So if you have a high edge probability, then typically, you would expect to see a triangle in a random instantiation of this random graph G(n,p).

One way to summarize both statements is the following. One over n is a threshold for the property of containing a triangle. To interpret the sentence, imagine if p is much larger than one over n. Then we're in the theorem 1 setting. Then, the random graph typically contains a triangle.

Whereas if p is much smaller than 1 over n in the sense of being in the theorem 0 setting, then typically, the random graph does not contain any triangles. So 1 over n is a threshold. Depending on whether the edge probability p is much larger or much smaller than 1 over n. we expect very different behaviors from the perspective of whether the graph contains a triangle.

These types of statements are widely studied in combinatorics and probability and understanding how to establish and prove these type of statements are some of the bread and butter of probabilistic combinatorics. So in this video, let us prove the statement in this box.

Let's start with the easier direction, which, in this case, is the first statement, theorem 0. So let's prove theorem 0. We're looking at triangles. So let us define a random variable, x, to be the number of triangles in the random graph G(n,p).

We can compute the expectation of x with not much difficulty, because by linearity of expectations, there are n choose three triples of vertices. And each triple forms a triangle with probability p cubed.

We see that this quantity here is on the order of n cubed p cubed. We get to drop constant factors in this asymptotic estimate. Now when (n,p) goes to zero, as n goes to infinity, n cubed, p cubed also goes to zero. So this quantity is little o of 1. It's a quantity that goes to 0 as n goes to infinity.

Let us now apply Markov's inequality. We explained and proved Markov's inequality in a different video. By Markov's inequality, we find that the probability that x is at least one is, at most, the expectation of x divided by 1. And from earlier, this quantity is little o of 1.

All right. So the number of triangles is typically small in expectation. So it's small in expectation. And therefore, the probability that there is at least one triangle goes to 0 as n goes to infinity. Thus, the previous statement is equivalent to the statement that the probability that there are zero triangles approaches 1 as n goes to infinity.

This finishes the first part of the theorem. In the first part, the edge probability is very small, and so the expected number of triangles goes to 0 as n goes to infinity. And by Markov's inequality, we see that typically, there should not be any triangles in this random graph.

Let us now move on to the second part. In the second part, namely theorem one, if we use the setup that we had earlier, we see that the number of triangles in expectation goes to infinity as n goes to infinity. However, this does not immediately imply that x is typically positive.

Because it could be the case that x is still typically zero, but is very, very large a diminishing fraction of the time. And in that case, we would still be able to have the statement that the expectation of x goes to infinity while it would not be true that x is typically positive. So we'll need additional ideas to establish that x is typically positive. And what we will do is to use the second moment bound to show that x is typically close to its mean.

So the x is very concentrated around its expectation. And in that case, having expectation that goes to infinity would then imply that x is typically positive. So let's carry out this idea.

Let's consider the following probability. We see that the probability that x is equal to 0. Well, if x is equal to 0, and here x is some non-negative quantity, then this statement is equivalent to the statement that x deviates from this mean in the downward direction by minus the expectation of x.

So this is the same as x. Less than equal to 0, but x is always non-negative. So these two probabilities are equal to each other. We can then relax the event inside as such. Again, noting that the expectation of x is always non-negative.

Now we apply a Chebyshev's inequality. So in this step, we can apply Chebyshev's inequality. And you'll find an explanation and proof of Chebyshev's inequality in a different video, which allows us to conclude that this probability is, at most, the variance of x divided by the expectation of x squared.

All right. So what have we learned from this calculation? Let me state a corollary. If the expectation of x is much smaller, so if the variance of x is much smaller than the square of expectation, then the right-hand side goes to 0 as n goes to infinity, which then implies that x is positive with high probability, meaning that this probability that x is positive goes to 1 as n goes to infinity.

So this is the conclusion that we will use to establish theorem 1. So the number of triangles, we'll be able to write in the following way. x is equal to the sum over all triples of vertices of the following quantity, xij, xik, xjk, where we define these x sub ij's as follows.

x sub ij is equal to 1 if ij is an edge and zero if ij is not an edge. The vertices are labeled by numbers 1 through n. So you see this sum, which is indexed over triples of vertices, each term is 1 if these vertices i, j, and k is a triangle and is 0 otherwise. So this is the number of triangles in the graph.

And let me, just because it will make the notation a little easier, rewrite this quantity as such. Here, t ranges over all triples of vertices. Onward the triples of vertices, and x sub t is this term so we can rewrite this term as just a single variable x sub ijk.

To apply the corollary, we will need to get some estimate from the variance of x. And we'll do so through covariance. So as a reminder, the covariance of two random variables, say y and z, is defined to be the expectation of y times z minus the expectation of y times the expectation of z.

In particular, if y and z are independent random variables, then their covariance is 0. And the variance of a random variable x is equal to the covariance of x with itself.

So let us compute the variance of x, x being the number of triangles. We can rewrite it as a covariance of x with itself. Here, x, we can write it as a sum of various terms indexed by triples of vertices.

I'll use t for the first index in the first sum and t prime for the indices in the second sum. The nice thing about covariance is that it is a bilinear function. If you split y in the covariance of y and z, if you split y into a linear combination of various sums, then you'll see that the dependence in this formula is linear on splitting y, so you will be able to write the covariance correspondingly as a sum.

So given this covariance of these two sums, we can distribute this quantity and write it as the covariance over all pairs of triples T and T prime, the covariance of x sub T against x sub T prime.

Now we need to understand this term, the covariance of x sub T against the covariance of x sub T prime. Towards this end, let us think about what happens to this term. So let me do a side calculation.

From the definition of covariance, we see that this quantity equals to the expectation of x sub T, x sub T prime minus the expectation of x sub T times the expectation of x sub T prime. There are a few cases that we need to consider.

First, T and T prime are two triples of vertices and they could be in various positions. The corresponding full triangles could be disjoint, or they could overlap. And we need to consider all of those possibilities.

Now before doing those cases, let's observe that this quantity here, if you have, for example, t and t prime being in any number of configurations. So these are the vertices of T and this is the vertices of T prime, for instance. So they could share some edges or they may overlap in some edges.

But no matter what the configurations are, the first term always equals to p raised to the number of edges contained in the union of T and T prime. So in the complete graph. So what I mean here is the following.

So here is p raised to the number of-- so p raised to-- let me still write it this way and then we'll see some specific examples. Minus p raised to the number of edges in T and p raised to the number of edges in t prime. So here, the exponents over correspond to the number of edges in the complete graph spanned by these vertices, T and T prime.

So let's look at a few different possibilities. The first is if the two triangles, T and T prime, do not intersect in any edges. So the number of vertices they intersect is at most one. So the two scenarios are the one that I just drew and if these two triangles are even disjoint in their vertex set.

So then there's no edge overlap, in which case the edges-- so the random variables that correspond to the edges of these two triangles, they are independent of each other. And so the covariance should be 0. And indeed, the first term here, you have p raised to the power 6 and then minus p cubed and p cubed. And so in this case, this covariance is 0.

In the second case, when these two triples T and T prime overlapping exactly two vertices. Well, here they are-- so the union of these two triangles-- so I think I misspoke earlier. So by this notation, I really mean viewing T as a triangle and T prime as a triangle, and looking at the union of these two triangles.

So the union of these two triangles have five edges. So that's the number of terms here, dismissing repetition in the first term, the number of factors. And minus the later two terms are both p cubed and p cubed. Finally, if T and T prime are identical, then the number of edges in the first term is 3 minus p cubed p cubed.

So recall that p is some quantity that goes to 0 as-- well, it doesn't have to go to 0, but we-- p is some quantity that for now you can think of as going to 0 so that the first term is always dominant. Even if p is constant up to a constant factor, still, the first term is dominant.

So let us now continue the earlier conversation-- the earlier computation. And we see that this sum is equal to the following. There are-- we need to think about how each of these cases can arise. The first case contributes zero to the sum, so we don't have to worry about it.

The second term, the second possibility contributes the following. It contributes n choose 2. Well, we need to count how many different configurations can we have like this, where you have these two, the red and the blue triangles overlapping by an edge.

So this n choose 2 for the two overlapping vertices and n times n minus-- times n minus 2 Times N minus 3 for the remaining two vertices. And in this case, we have p to the 5 minus p to the 6. And then the third possibility, n choose 3 times p cubed minus p to the sixth power.

This is some expression, and we only need to get some bounds so we can play around with asymptotics and not think about constant factors. So we'll basically ignore the constant factors and do an upper bound. So up to a constant factor. So big O.

The first term here involves n to the fourth p fifth power. So we can ignore the negative contributions. They don't matter for this calculation. And then the second term here is n cubed p cubed. At most, on the order of n cubed p cubed.

So under the hypothesis that np goes to infinity, Both these terms are little o of n to the six. p to the sixth. So that's something that one can check from the from the hypothesis. Thus, the variance of x is little o of the expectation of x squared.

And so by the corollary of Chebyshev earlier, we see that with probability approaching 1, x is positive. So that there's-- typically, this random graph contains a triangle. And that finishes the proof of the second part, this theorem 1.

So to recap, we showed that 1 over n is a threshold for the property of the random graph G(n,p) to contain a triangle. So the statement has two parts. The first part is that when p is quite small, it's much smaller than 1 over n, then typically, this random graph does not contain a triangle.

And the way we establish this fact is by showing that the number of triangles, x, the number of triangles, has expected value going to 0. And thus, by Markov's inequality, this number of triangles is 0 with high probability. The second part, where p now is much larger than 1 over n, here, the expected number of triangles goes to infinity as n goes to infinity. But that alone does not allow us to conclude that the number of triangles is positive typically.

Instead, we'll need to apply a second moment argument to show that the number of triangles is typically concentrated around its mean. And that was a calculation that we did. This was a second moment calculation, namely understanding and computing the variance of this random variable and showing that the variance is much smaller than the square of the expectation from which then we can use Chebyshev's inequality to deduce that the number of triangles is typically positive. And combining these two proofs together, we obtain this threshold statement.

Now what we just showed is an important, but fairly basic application of using probabilistic methods to understand typicality statements. And these types of statements and proof techniques are quite ubiquitous in studies involving the probabilistic method and beyond. And there are much more difficult results of similar flavor, but that involve much more advanced techniques. And what you saw here is simply a taste of what such a statement and its technique could look like.