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YUFEI ZHAO: In this video, we'll look at an application of the probabilistic method to extremal set theory. Extremal set theory is an area of combinatorics concerned with questions such as, what is the largest collection of sets that satisfy certain desired properties?

And the question that we'll look at in this video is the following. What is the largest antichain? And more precisely, what is the largest collection of subsets of some ground set numbers 1 through n-- so this is an n element set. So I want as many sets as we can such that none of these sets is a subset of another set. So this is what we mean by an antichain. And for a given value of n, what is the largest number of sets you can have? What is the largest l that we can have as a function of n?

To give an example, when n equals to 3, you can have the sets 1, 2; 1, 3; and 2, 3; and you see that none of these sets is a superset of another. More generally, when you take all k-element subsets of 1 through n, this collection also has the property that no set is a subset of another set because all these k-element subsets have the same size, so none of them can contain another.

Now this collection of sets have n choose k sets. And which value of k maximizes this quantity n choose k? Well, it is maximized at k being n over 2, or when n is odd, we can either round down or round up, leading to n choose k over 2, or the floor of k over 2-- so this is-- so n choose n over 2. So n over 2 floor, meaning that's rounded down or, equivalently, rounded up, will produce the same answer. So this many sets we can get from an antichain.

Now this is just one example. And the question is, is this the best example? Can we do better? Can we get an even larger collection of sets that form an antichain? And the classic result of Sperner's-- or Sperner's theorem, such that-- actually, no, this is the best that we can do.

And what that theorem says is that if you have l subsets of 1 through n such that none of the sets is a subset of another, then, in fact, l is, at most, n choose the floor of n over 2. So this answers the question that we were asking. And in this video, we'll use the probabilistic method to prove Sperner's theorem.

And it's quite a beautiful proof. And the introduction of randomness into the proof is quite remarkable for a theorem that really doesn't involve any randomness at all.

What we will prove is actually the following theorem, which is slightly stronger than Sperner's theorem. And we'll prove the following, also known as the LYM inequality, named after three people who discovered the theorem. And it says that in the hypothesis that we just set up, we furthermore have the following consequence.

If you take the sum over I from 1i-- i being 1 to l of the quantity 1 over n choose the size of Ai, then this sum is, at most, 1. So let us first prove this LYM inequality, and then afterwards, we'll deduce Sperner's theorem from this LYM inequality.

Let's now introduce randomness into the problem. So let sigma 1 through sigma n be a randomly chosen permutation. of numbers 1 through n. And here, we'll choose this permutation uniformly at random, meaning that of all the n factorial different permutations, we'll select one with equal probability of being any one of those n factorial permutations.

And now let's consider the following chain of sets, starting with the antiset, and then the set containing just the first element of this permutation, sigma 1, and then the set containing the first two elements of this permutation, sigma 1, sigma 2, and so on, until we add all the elements into the set. So these are the set of prefixes of this permutation where we add one element at a time. So this is a chain of subsets of 1 through n.

And let's consider the event for which we'll evaluate its probability. So the event that Ai-- so Ai is one of the sets that was originally given in the in the theorem statement. So what's the event? So the event that we'll consider is the event that Ai appears in this chain. So this chain means this chain over here.

Now, this chain is random because the sigmas are a random permutation. Even though the Ai's are deterministic, the Ai's are given, this permutation is random. So this is some event which has a probability.

When do we have Ai appearing in this chain? Or if all the elements of AI appear in this permutation before all the non-elements of Ai. So we can count how many different ways this can happen. So if all the n factorial are different permutations, the number of permutations where the elements of Ai appear first is Ai size factorial, and then all the non-elements of Ai appear afterwards. It's n minus the size of Ai factorial. And this quantity here is equal to 1 over the binomial coefficient n choose the size of Ai.

OK. Next, let us note the following. That no two different Ai's, Aj's can simultaneously appear in the chain. Why is that? Well, we assumed that no Ai contains is a subset of another Aj, and if, indeed, you had Ai and Aj both appearing in the chain, then one of them would contain another. So this chain cannot contain two different Ai's at the same time.

And therefore, these events, meaning these events over here, are disjoint as we run i from 1 through l. If these events are disjoint and each of them have this probability, well, the sum of these probabilities then must add up to, at most, 1.

And that's indeed the conclusion that we're looking for. So the sum of these probabilities, the probability that Ai appears in the chain, in this random chain, the sum of these probabilities add up to, at most, 1 because these events are disjoint from each other. And we have calculated earlier that the event probabilities are given as 1 over n choose the size of Ai.

So this concludes the proof of the LYM inequality. You see the final line, the final inequality is precisely what we were trying to establish. And finally, let us prove Sperner's theorem by deducing this theorem from the LYM inequality.

So here is a-- this is a quick deduction because n choose the size of Ai is, at most, n choose the floor of n over 2 for all i. So the binomial coefficients for a given n maxes out in the middle, we have this inequality. And thus-- here, I'm just rewriting the LYM inequality. The sum from-- so i from one to l of 1 over n choose size of Ai. This is, at most, 1, so that's what we just proved.

And then applying this individual bound to each term, we get l different terms, and each one of them in the denominator, we have n choose the floor of n over 2. And rearranging this inequality, we get the desired upper bound on l.

So this concludes the proof of Sperner's theorem. And it's a beautiful result with a beautiful proof that is a wonderful illustration of the probabilistic method in combinatorics where, although the theorem statement itself is deterministic-- doesn't involve any randomness, the proof works by introducing randomness into the problem. So this is the application of the probabilistic method in combinatorics, and I think it's one of the most beautiful applications.