9. Fermionic integrals

9.1. **Bosons and fermions.** In physics there exist two kinds of particles – bosons and fermions. So far we have dealt with bosons only, but many important particles are fermions: e.g., electron, proton, etc. Thus it is important to adapt our techniques to the fermionic case.

In quantum theory, the difference between bosons and fermions is as follows: if the space of states of a single particle is $\mathcal{H}$ then the space of states of the system of k such particles is $S^k\mathcal{H}$ for bosons and $\Lambda^k\mathcal{H}$ for fermions. In particular, in the fermionic case, if $\dim \mathcal{H} = n$ then the space of states of $\geq n+1$ identical particles is zero, which is the *Pauli exclusion principle* (leading, for instance, to the fact that the number of electrons in an atom at the m-th energy level is bounded by $2m^2$). In classical theory, this means that the space of states of a bosonic particle is a usual real vector space (or, more generally, a manifold), while for a fermionic particle it is an *odd vector space*. Mathematically "odd" means that the algebra of smooth functions on this space (i.e. the algebra of classical observables) is an *exterior* algebra (unlike the case of a usual, *even* space, for which the algebra of polynomial functions is a *symmetric* algebra).

More generally, one may consider systems of classical particles or fields some of which are bosonic and some fermionic. In this case, the space of states will be a supervector space, i.e. the direct sum of an even and an odd space (or, more generally, a supermanifold – a notion we will define below).

When such a theory is quantized using the path integral approach, one has to integrate functions over supermanifolds. Thus, we should learn to integrate over supermanifolds and then generalize to this case our Feynman diagram techniques. This is what we do in this section.

9.2. **Supervector spaces.** Let k be a field of characteristic zero. A *supervector space* (or shortly, superspace) over k is just a $\mathbb{Z}/2$-graded vector space: $V = V_0 \oplus V_1$. If $V_0 = k^n$ and $V_1 = k^m$ then V is denoted by $k^{n|m}$. The notions of a linear operator, direct sum, tensor product, dual space for supervector spaces are defined in the same way as for $\mathbb{Z}/2$-graded vector spaces. In other words, the tensor category of supervector spaces is the same as that of $\mathbb{Z}/2$-graded vector spaces.

However, the notions of a supervector space and a $\mathbb{Z}/2$-graded vector space are *not* the same. The difference is as follows. The category of vector (and hence $\mathbb{Z}/2$-graded vector) spaces has a *symmetric structure*, which is the standard isomorphism $V \otimes W \to W \otimes V$ (given by $v \otimes w \to w \otimes v$). This isomorphism allows one to define symmetric powers S^iV, exterior powers Λ^iV, etc. For supervector spaces, there is also a symmetry $V \otimes W \to W \otimes V$, but it is defined differently. Namely, $v \otimes w$ goes to $(-1)^{ij} w \otimes v$, $v \in V_i, w \in V_j$ $(i, j \in \{0, 1\})$. In other words, it is the same as usual except that if v, w are both odd then $v \otimes w \mapsto -w \otimes v$. As a result, we can define the superspaces $S^i V$ and $\Lambda^i V$ for a superspace V, but they are not the same as the symmetric and exterior powers in the usual sense. For example, if V is purely odd ($V = V_1$), then $S^i V$ is the i-th exterior power of V, and $\Lambda^i V$ is the i-th symmetric power of V (purely even for even i and purely odd for odd i). Thus in general for $V = V_0 \oplus V_1$, we have the following expressions for the symmetric algebra $SV := \oplus_{i \geq 0} S^i V$ and exterior algebra $\Lambda V := \oplus_{i \geq 0} \Lambda^i V$:

$$SV = SV_0 \otimes \Lambda V_1, \ \Lambda V = \Lambda V_0 \otimes SV_1.$$

For a superspace V, let ΠV be the same space with opposite parity, i.e. $(\Pi V)_j = V_{1-j}$, $j = 0, 1$. Then we have

$$S^i V = \Pi^i (\Lambda^i \Pi V), \ \Lambda^i V = \Pi^i (S^i \Pi V).$$

Let $V = V_0 \oplus V_1$ be a finite dimensional superspace. Define the algebra of polynomial functions on V, $\mathcal{O}(V)$, to be the algebra SV^* (where symmetric powers are taken in the supersense). Thus, $\mathcal{O}(V) = SV_0^* \otimes \Lambda V_1^*$, where V_0 and V_1 are regarded as usual spaces. More explicitly, if $x_1, ..., x_n$ are linear coordinates on V_0, and $\xi_1, ..., \xi_m$ are linear coordinates on V_1, then $\mathcal{O}(V) = k[x_1, ..., x_n, \xi_1, ..., \xi_m]$, with defining relations

$$x_i x_j = x_j x_i, \ x_i \xi_r = \xi_r x_i, \ \xi_r \xi_s = -\xi_s \xi_r$$

(in particular, $\xi_r^2 = 0$). Note that this algebra is itself a (generally, infinite dimensional) supervector space, and is commutative in the supersense. Also, if V, W are two superspaces, then $\mathcal{O}(V \oplus W) = \mathcal{O}(V) \otimes \mathcal{O}(W)$, where the tensor product of algebras is understood in the supersense, i.e.

$$(a \otimes b)(c \otimes d) = (-1)^{p(b)p(c)} (ac \otimes bd),$$

where $p(x)$ is the parity of x.

9.3. **Supermanifolds.** Now assume that $k = \mathbb{R}$. Then by analogy with the above for any supervector space V we can define the algebra of smooth functions, $C^\infty(V) := C^\infty(V_0) \otimes \Lambda V_1^*$. In fact, this is a special case of the following more general setting.

Definition 9.1. A *supermanifold* M is a usual manifold M_0 with a sheaf C_M^∞ of $\mathbb{Z}/2\mathbb{Z}$ graded algebras (called the *structure sheaf*), which is locally isomorphic to $C_{M_0}^\infty \otimes \Lambda(\xi_1, ..., \xi_m)$.

The manifold M_0 is called the *reduced manifold* of M. The dimension of M is the pair of integers $\dim M_0|m$.

For example, a supervector space V is a supermanifold of dimension $\dim V_0|\dim V_1$. Another (more general) example of a supermanifold is a superdomain $U := U_0 \times V_1$, i.e. a domain $U_0 \subset V_0$ together with the sheaf $C^\infty_{U_0} \otimes \Lambda V_1^*$. Moreover, the definition of a supermanifold implies that any supermanifold is "locally isomorphic" to a superdomain.

Let M be a supermanifold. An *open set* U in M is the supermanifold $(U_0, C^\infty_M|_{U_0})$, where U_0 is an open subset in M_0.

By the definition, supermanifolds form a category $\mathcal{S}$. Let us describe explicitly morphisms in this category, i.e. maps $F : M \to N$ between supermanifolds M and N. By the definition, it suffices to assume that M, N are superdomains, with global coordinates $x_1, ..., x_n, \xi_1, ..., \xi_m$, and $y_1, ..., y_p, \eta_1, ..., \eta_q$, respectively (here x_i, y_i are even variables, and ξ_i, η_i are odd variables). Then the map F is defined by the formulas:

$$y_i = f_{0,i}(x_1, ..., x_n) + f_{2,i}^{j_1 j_2}(x_1, ..., x_n)\xi_{j_1}\xi_{j_2} + ...,$$

$$\eta_i = a_{1,i}^{j}(x_1, ..., x_n)\xi_j + a_{3,i}^{j_1 j_2 j_3}(x_1, ..., x_n)\xi_{j_1}\xi_{j_2}\xi_{j_3} + ...,$$

where $f_{0,i}, f_{2,i}^{j_1 j_2}, ..., a_{1,i}^{j}, a_{3,i}^{j_1 j_2 j_3}, ...$ are usual smooth functions, and we assume summation over repeated indices. These formulas, determine F completely, since for any $g \in C^\infty(N)$ one can find $g \circ F \in C^\infty(M)$ by Taylor's formula. For example, if $M = N = \mathbb{R}^{1|2}$, $F(x, \xi_1, \xi_2) = (x + \xi_1\xi_2, \xi_1, \xi_2)$, and $g = g(x)$, then

$$g \circ F(x, \xi_1, \xi_2) = g(x + \xi_1\xi_2) = g(x) + g'(x)\xi_1\xi_2.$$

9.4. **Supermanifolds and vector bundles.** Let M_0 be a manifold, and E be a real vector bundle on M_0. Then we can define the supermanifold $M := \mathrm{Tot}(\Pi E)$, the total space of E with changed parity. Namely, the reduced manifold of M is M_0, and the structure sheaf C^∞_M is the sheaf of sections of ΛE^*. This defines a functor $S : \mathcal{B} \to \mathcal{S}$, from the category of manifolds with vector bundles to the category of supermanifolds. We also have a functor S_* in the opposite direction: namely, $S_*(M)$ is the manifold M_0 with the vector bundle $(R/R^2)^*$, where R is the nilpotent radical of C^∞_M.

The following proposition (whose proof we leave as an exercise) gives a classification of supermanifolds.

Proposition 9.2. *(i)* $S_* \circ S = \mathrm{Id}$*;*

(ii) $S \circ S_* = \mathrm{Id}$ *on isomorphism classes of objects.*

The usefulness of this proposition is limited by the fact that, as one can see from the above description of maps between supermanifolds,

$S \circ S_*$ is *not* the identity on morphisms (e.g. it maps the automorphism $x \to x + \xi_1\xi_2$ of $\mathbb{R}^{1|2}$ to Id), and hence, S is not an equivalence of categories. In fact, the category of supermanifolds is not equivalent to the category of manifolds with vector bundles (namely, the category of supermanifolds "has more morphisms").

Remark 9.3. 1. The relationship between these two categories is quite similar to the relationship between the categories of (finite dimensional) filtered and graded vector spaces, respectively (namely, for them we also have functors S, S_* with the same properties – check it!). Therefore in supergeometry, it is better to avoid using realizations of supermanifolds as $S(M_0, E)$, similarly to how in linear algebra it is better to avoid choosing a splitting of a filtered space.

2. In the definition of a supermanifold one can replace the real exterior algebra $\Lambda(\xi_1, ..., \xi_m)$ with the complexified exterior algebra $\Lambda_{\mathbb{C}}(\xi_1, ..., \xi_m)$. This gives a notion of a $\mathbb{C}$-supermanifold, which generalizes the notion of an ordinary smooth manifold with the sheaf of complex-valued (as opposed to real-valued) smooth functions. Similarly to Proposition 9.2, isomorphism classes of $\mathbb{C}$-supermanifolds with reduced submanifolds M_0 are in bijection with isomorphism classes of *complex* vector bundles on M_0, so they are more general (as not every complex vector bundle is the complexification of a real one). Otherwise, the theory of $\mathbb{C}$-supermanifolds (which does actually arise in quantum field theory, see Remark 11.3 below) is completely parallel to the theory of usual supermanifolds.

One may also similarly define complex analytic and algebraic supermanifolds, but this is a different story which we will not discuss here.

9.5. **Supertrace and superdeterminant (Berezinian).** Before proceeding further, we need to generalize to the supercase the basic notions of linear algebra, such as trace and determinant of a matrix.

Let $R := R_0 \oplus R_1$ be a supercommutative $\mathbb{C}$-algebra. Fix two nonnegative integers m, n. Let $\mathrm{Mat}_{n|m}(R)$ be the algebra of $n+m$ by $n+m$ matrices over R which have the block decomposition

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

so that A_{00} is n by n, A_{11} is m by m, and A_{00}, A_{11} have even entries (i.e., in R_0), while A_{01}, A_{10} have odd entries (i.e., in R_1). We would like to define the *supertrace* of A as a linear function

$$\mathrm{sTr}(A) = \sum_{i,j=1}^{n+m} \lambda_{ij} a_{ij}, \ \ \lambda_{ij} \in \mathbb{Z},$$

so that $\mathrm{sTr}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = n$ and $\mathrm{sTr}(AB) = \mathrm{sTr}(BA)$ for any R and $A, B \in \mathrm{Mat}_{n|m}(R)$. Thus we must have $\mathrm{sTr}(A) = \mathrm{Tr}(A_{00}) + \varepsilon\mathrm{Tr}(A_{11})$ for some $\varepsilon \in \mathbb{Z}$, and taking all blocks of A, B except A_{01}, B_{10} to be zero, we get $\varepsilon = -1$. So the supertrace of A has to be defined by the formula

$$\mathrm{sTr}(A) = \mathrm{Tr}(A_{00}) - \mathrm{Tr}(A_{11}).$$

Now let us generalize to the supercase the definition of determinant. For a finite dimensional algebra R and $C \in \mathrm{Mat}_{n|m}(\mathbb{R})$ we would like to have

$$\mathrm{sdet}(e^C) = e^{\mathrm{sTr}C} = e^{\mathrm{Tr}(C_{00}) - \mathrm{Tr}(C_{11})}, \tag{9.1}$$

which generalizes the usual property of trace and determinant. So in the case of a block-diagonal matrix $C = C_{00} \oplus C_{11}$ we get

$$\mathrm{sdet}(e^C) = \frac{\det(e^{C_{00}})}{\det(e^{C_{11}})}.$$

Thus if $A = A_{00} \oplus A_{11}$ is block-diagonal, we must have

$$\mathrm{sdet}A = \frac{\det A_{00}}{\det A_{11}}.$$

This shows that we cannot hope that the superdeterminant will be a polynomial in the entries of A – it has to be a rational function defined only on some open subset. In fact, if we want to have the usual property $\mathrm{sdet}(AB) = \mathrm{sdet}(A)\mathrm{sdet}(B)$ then there is just one possibility. Indeed, suppose that

$$A = \begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}a_+ & 0\\ 0 & a_-\end{pmatrix}\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix} = \begin{pmatrix}a_+ + ba_-c & ba_-\\ a_-c & a_-\end{pmatrix}.$$

By (9.1), we must have

$$\mathrm{sdet}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} = \mathrm{sdet}\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix} = 1,$$

hence

$$\mathrm{sdet}(A) = \frac{\det a_+}{\det a_-}.$$

In other words, the superdeterminant has to be defined by the formula

$$\mathrm{sdet}(A) = \frac{\det(A_{00} - A_{01}A_{11}^{-1}A_{10})}{\det(A_{11})}$$

provided that A_{11} is invertible; otherwise the superdeterminant is not defined.

This function is also called the *Berezinian* of A and denoted $\mathrm{Ber}(A)$. So for $m=0$ one has $\mathrm{Ber}(A)=\det(A)$, and for $n=0$ one has $\mathrm{Ber}(A)=(\det A)^{-1}$.

Remark 9.4. Recall for comparison that if A is a purely even block matrix then

$$\det(A)=\det(A_{00}-A_{01}A_{11}^{-1}A_{10})\det(A_{11}).$$

Proposition 9.5. *(i) For any $A,B\in \mathrm{Mat}_{n|m}(R)$ with A_{11},B_{11} invertible, we have*

$$\mathrm{Ber}(AB)=\mathrm{Ber}(A)\mathrm{Ber}(B).$$

(ii) If R is finite dimensional and $A(t)\in \mathrm{Mat}_{n|m}(R)$ is a C^1-function near 0 with $A(0)$ invertible then

$$\tfrac{d}{dt}|_{t=0}\mathrm{Ber}(A(t))=\mathrm{sTr}(A'(0)A(0)^{-1})\mathrm{Ber}(A(0)).$$

(iii) If R is finite dimensional then for any $C\in \mathrm{Mat}_{n|m}(R)$ we have

$$\mathrm{Ber}(e^C)=e^{\mathrm{sTr}C}.$$

Proof. (i) From the triangular factorization, it is clear that it suffices to consider the case

$$A=\begin{pmatrix}1&0\\X&1\end{pmatrix},\ B=\begin{pmatrix}1&Y\\0&1\end{pmatrix},$$

where X,Y are matrices with odd elements, so that

$$AB=\begin{pmatrix}1&Y\\X&1+XY\end{pmatrix}.$$

Then the required identity is

$$\det(1-Y(1+XY)^{-1}X)=\det(1+XY).$$

To prove this identity, recall that $X:V_0\to V_1\otimes R$ and $Y:V_1\to V_0\otimes R$. We have

$$\det(1-Y(1+XY)^{-1}X)=\sum_{k\ge 0}(-1)^k\mathrm{Tr}(Y(1+XY)^{-1}X|_{\Lambda^kV_0})=$$

$$=\sum_{k\ge 0}(-1)^k\mathrm{sTr}(Y(1+XY)^{-1}|_{\Lambda^kV_1}\circ X|_{\Lambda^kV_0})=\sum_{k\ge 0}(-1)^k\mathrm{sTr}(XY(1+XY)^{-1}|_{\Lambda^kV_1})$$

$$\sum_{k\ge 0}\mathrm{Tr}(XY(1+XY)^{-1}|_{S^k\Pi V_1})=\det(1-XY(1+XY)^{-1})^{-1}=\det(1+XY).$$

(ii) By (i) we may replace $A(t)$ by $A(t)A(0)^{-1}$, so it suffices to consider the case $A(0)=1$, where the statement easily follows from the definition.

(iii) Consider the function $f(t):=\mathrm{Ber}(e^{Ct})$. By (ii) it satisfies the differential equation $f'(t)=\mathrm{sTr}(C)f(t)$ with $f(0)=1$. Thus $f(t)=e^{\mathrm{sTr}(C)t}$, and the statement follows by setting $t=1$. □

9.6. **Integration on superdomains.** We would now like to develop integration theory on supermanifolds. Before doing so, let us recall how it is done for usual manifolds. In this case, one proceeds as follows.

1. Define integration of compactly supported (say, smooth) functions on a domain in $\mathbb{R}^n$.

2. Find the transformation formula for the integral under change of coordinates (i.e. discover the factor $|J|$, where J is the Jacobian).

3. Define a *density* on a manifold to be a quantity which is locally the same as a function, but multiplies by $|J|$ under coordinate change (unlike true functions, which don't multiply by anything). Then define integral of compactly supported densities on the manifold using partitions of unity. The independence of the integral on the choices is guaranteed by the change of variable formula and the definition of a density.

We will now realize this program for supermanifolds. We start with defining integration over superdomains.

Let $V = V_0 \oplus V_1$ be a supervector space. The *Berezinian* of V is the line $\Lambda^{\rm top} V_0^* \otimes \Lambda^{\rm top} V_1$ (where V_0, V_1 are treated as usual spaces). Suppose that V is equipped with a nonzero element dv of its Berezinian (called a *supervolume element*).

Let U_0 be an open set in V_0, and $f \in C^\infty(U_0) \otimes \Lambda V_1^*$ be a compactly supported smooth function on the superdomain $U := U_0 \times V_1$ (i.e. $f = \sum f_i \otimes \omega_i$, $f_i \in C^\infty(U_0)$, $\omega_i \in \Lambda V_1^*$, and f_i are compactly supported). Let dv_0, dv_1 be volume forms on V_0, V_1 such that $dv = dv_0/dv_1$.

Definition 9.6. The integral $\int_U f(v)dv$ is $\int_{U_0} (f(v), (dv_1)^{-1}) dv_0$.

It is clear that this quantity depends only on dv and not on dv_0 and dv_1 separately.

Thus, $\int_U f(v)dv$ is defined as the integral of the suitably normalized top coefficient of f (expanded with respect to some homogeneous basis of ΛV_1^*). To write it in coordinates, let $x_1, ..., x_n, \xi_1, ..., \xi_m$ be a linear system of coordinates on V such that $dv = \frac{dx_1...dx_n}{d\xi_1...d\xi_m}$ (such coordinate systems will be called unimodular with respect to dv). Then $\int_U f(v)dv$ equals $\int_{U_0} f_{\rm top}(x_1, ..., x_n) dx_1...dx_n$, where $f_{\rm top}$ is the coefficient of $\xi_1...\xi_m$ in the expansion of f.

9.7. **Berezin's change of variable formula.** Let V be a vector space, $f \in \Lambda V^*$, $v \in V$. Denote by $\frac{\partial f}{\partial v}$ the result of contraction of f with v.

Let U, U' be superdomains, and $F : U \to U'$ be a morphism. As explained above, given linear coordinates $x_1, ..., x_n, \xi_1, ..., \xi_m$ on U and $y_1, ..., y_p, \eta_1, ..., \eta_q$ on U', we can describe F by expressing y_i and η_j as functions of x_i and ξ_j. Define the *Berezin matrix* of F, $A := DF(x,\xi)$ by the formulas:

$$A_{00} = (\frac{\partial y_i}{\partial x_k}),\ A_{01} = (\frac{\partial y_i}{\partial \xi_\ell}),\ A_{10} = (\frac{\partial \eta_j}{\partial x_k}),\ A_{11} = (\frac{\partial \eta_j}{\partial \xi_\ell}).$$

Clearly, this is a superanalog of the Jacobi matrix.

The main theorem of supercalculus is the following theorem.

Theorem 9.7. *(Berezin) Let g be a smooth function with compact support on U', and $F : U \to U'$ be an isomorphism. Let dv, dv' be supervolume elements on U, U'. Then*

$$\int_{U'} g(v')dv' = \int_U g(F(v))|\mathrm{Ber}(DF(v))|dv,$$

where the Berezinian is computed with respect to unimodular coordinate systems.

Here if $f(\xi) = a+$terms containing ξ_j, $a \in \mathbb{R}$, $a \neq 0$ then by definition $|f(\xi)| := f(\xi)$ is $a > 0$ and $|f(\xi)| := -f(\xi)$ if $a < 0$.

Proof. The chain rule of the usual calculus extends verbatim to supercalculus. Thus, since $\mathrm{Ber}(AB) = \mathrm{Ber}(A)\mathrm{Ber}(B)$, if we know the statement for two isomorphisms $F_1 : U_2 \to U_1$ and $F_2 : U_3 \to U_2$, then we know it for the composition $F_1 \circ F_2$.

Let $F(x_1, ..., x_n, \xi_1, ..., \xi_m) = (x'_1, ..., x'_n, \xi'_1, ..., \xi'_m)$. We see that it suffices to consider the following cases.

1. x'_i depend only on x_k, $k = 1, ..., n$, and $\xi'_j = \xi_j$.
2. $x'_i = x_i + z_i$, where z_i lie in the ideal generated by ξ_j, and $\xi'_j = \xi_j$.
3. $x'_i = x_i$.

Indeed, it is clear that any isomorphism F is a composition of isomorphisms of types $1, 2, 3$.

In case 1, the statement of the theorem follows from the usual change of variable formula. Thus it suffices to consider cases 2 and 3.

In case 2, it is sufficient to consider the case when only one coordinate is changed by F, i.e. $x'_1 = x_1 + z$, and $x'_i = x_i$ for $i \geq 2$. In this case we have to show that the integral of

$$g(x_1 + z, x_2, ..., x_n, \xi)(1 + \frac{\partial z}{\partial x_1}) - g(x_1, x_2, ..., x_n, \xi)$$

is zero. But this follows easily upon expansion in powers of z, since all the terms are manifestly total derivatives with respect to x_1.

In case 3, we can also assume $\xi'_j = \xi_j$, $j \geq 2$, and a similar (actually, even simpler) argument proves the result. □

9.8. **Integration on supermanifolds.** Now we will define densities on supermanifolds. Let M be a supermanifold, and $\{U_\alpha\}$ be an open cover of M together with isomorphisms $f_\alpha : U_\alpha \to U'_\alpha$, where U'_α is a superdomain in $\mathbb{R}^{n|m}$. Let $g_{\alpha\beta} : f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$ be the transition map $f_\alpha f_\beta^{-1}$. Then a density s on M is a choice of an element $s_\alpha \in C_M^\infty(U_\alpha)$ for each α, such that on $U_\alpha \cap U_\beta$ one has $s_\beta(z) = s_\alpha(z)|\mathrm{Ber}(g_{\alpha\beta})(f_\beta(z))|$.

Remark 9.8. It is clear that a density on M is a global section of a certain sheaf on M, called the sheaf of densities.

Now, for any (compactly supported) density ω on M, the integral $\int_M \omega$ is well defined. Namely, it is defined as in usual calculus: one uses a partition of unity ϕ_α such that $\mathrm{Supp}\phi_\alpha \subset (U_\alpha)_0$ are compact subsets, and sets $\int_M \omega := \sum_\alpha \int_M \phi_\alpha \omega$ (where the summands can be defined using f_α). Berezin's theorem guarantees then that the final answer will be independent on the choices made.

9.9. **Gaussian integrals in an odd space.** Now let us generalize to the odd case the theory of Gaussian integrals, which was, in the even case, the basis for the path integral approach to quantum mechanics and field theory.

Recall first the notion of *Pfaffian*. Let A be a skew-symmetric matrix of even size. Then the determinant of A is the square of a polynomial in the entries of A. This polynomial is determined by this condition up to sign. The sign is usually fixed by requiring that the polynomial should be 1 for the direct sum of matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. With this convention, this polynomial is called the Pfaffian of A and denoted $\mathrm{Pf}A$. The Pfaffian obviously has the property $\mathrm{Pf}(X^T A X) = \mathrm{Pf}(A)\det(X)$ for any matrix X.

Let now V be a $2m$-dimensional vector space with a volume element dv, and B a skew-symmetric bilinear form on V. We define the Pfaffian $\mathrm{Pf}B$ of B to be the Pfaffian of the matrix of B in any unimodular basis (by the above transformation formula, it does not depend on the choice of the basis). It is easy to see (by reducing B to the canonical form) that

$$\frac{\Lambda^m B}{m!} = \mathrm{Pf}(B)dv.$$

In terms of matrices, this translates into the following (well known) formula for the Pfaffian of a skew symmetric matrix of size $2m$:

$$\mathrm{Pf}(A) = \sum_{\sigma \in \Pi_m} \varepsilon_\sigma \prod_{i \in \{1,\dots,2m\}, i<\sigma(i)} a_{i\sigma(i)},$$

where Π_m is the set of matchings of $\{1, ..., 2m\}$, and ε_σ is the sign of the permutation sending $1, ..., 2m$ to $i_1, \sigma(i_1), ..., i_m, \sigma(i_m)$ (where $i_r < \sigma(i_r)$ for all r). For example, for $m = 2$ (i.e. a 4 by 4 matrix),

$$\mathrm{Pf}(A) = a_{12}a_{34} + a_{14}a_{23} - a_{13}a_{24}.$$

Now consider an odd vector space V of dimension $2m$ with a volume element $d\xi$. Let B be a symmetric bilinear form on V (i.e. a skewsymmetric form on ΠV). Let $\xi_1, ..., \xi_{2m}$ be unimodular linear coordinates on V (i.e. $d\xi = d\xi_1 \wedge ... \wedge d\xi_{2m}$). So if $\xi = (\xi_1, ..., \xi_{2m})$ then $B(\xi, \xi) = \sum_{i,j} b_{ij}\xi_i\xi_j$, where b_{ij} is a skewsymmetric matrix.

Proposition 9.9.

$$\int_V e^{\frac{1}{2}B(\xi,\xi)}(d\xi)^{-1} = \mathrm{Pf}(B).$$

Proof. The integral equals $\frac{1}{m!}\frac{\wedge^m B}{d\xi}$, which is precisely $\mathrm{Pf}(B)$. □

This formula has the following important special case. Let Y be a finite dimensional odd vector space, and $V = Y \oplus Y^*$. The space Y has a canonical volume element $dv = dydy^*$, defined as follows: if $e_1, ..., e_m$ is a basis of Y and $e_1^*, ..., e_m^*$ is the dual basis of Y^* then $dydy^* = e_1 \wedge e_1^* \wedge ... \wedge e_n \wedge e_n^*$.

Let $A : Y \to Y$ be a linear operator. Then we can define an even smooth function S on the odd space Y as follows: $S(y, y^*) = (Ay, y^*)$. More explicitly, if ξ_i are coordinates on Y corresponding to the basis e_i, and η_i the dual system of coordinates on Y^*, then

$$S(\xi_1, ..., \xi_m, \eta_1, ..., \eta_m) = \sum_{i,j} a_{ij}\xi_j\eta_i,$$

where (a_{ij}) is the matrix of A in the basis e_i.

Proposition 9.10.

$$\int_V e^S (dv)^{-1} = (-1)^{\frac{n(n-1)}{2}} \det A.$$

Proof. We have $S(y, y_*) = \frac{1}{2}B((y, y_*), (y, y_*))$, where B is the skewsymmetric form on ΠV given by the formula

$$B((y, y^*), (w, w^*)) = (Ay, w^*) - (Aw, y^*).$$

It is easy to see that $\mathrm{Pf}(B) = (-1)^{\frac{n(n-1)}{2}} \det(A)$, so Proposition 9.10 follows from Proposition 9.9.

Another proof can be obtained by direct evaluation of the top coefficient. □

9.10. **The Wick formula in the odd case.** Let V be a $2m$-dimensional odd space with a volume form $d\xi$, and $B \in S^2V^*$ a non-degenerate form (symmetric in the supersense and antisymmetric in the usual sense). Let $\lambda_1, ..., \lambda_n$ be linear functions on V. Then $\lambda_1, ..., \lambda_n$ can be regarded as odd smooth functions on the superspace V.

Theorem 9.11.

$$\int_V \lambda_1(\xi)...\lambda_n(\xi)e^{-\frac{1}{2}B(\xi,\xi)}(d\xi)^{-1} = \mathrm{Pf}(-B)\mathrm{Pf}(B^{-1}(\lambda_i, \lambda_j)).$$

(By definition, this is zero if n is odd). In other words, we have:

$$\int_V \lambda_1(\xi)...\lambda_n(\xi)e^{-\frac{1}{2}B(\xi,\xi)}(d\xi)^{-1} =$$

$$\mathrm{Pf}(-B)\sum_{\sigma\in\Pi_m} \varepsilon_\sigma \prod_{i\in\{1,...,2m\}, i<\sigma(i)} B^{-1}(\lambda_i, \lambda_{\sigma(i)}).$$

Proof. We prove the second formula. Choose a basis e_i of V with respect to which the form B is standard: $B(e_j, e_l) = 1$ if $j = 2i-1, l = 2i$, and $B(e_j, e_l) = 0$ for other pairs $j < l$. Since both sides of the formula are polylinear with respect to $\lambda_1, ..., \lambda_n$, it suffices to check it if $\lambda_1 = e^*_{i_1}$,..., $\lambda_n = e^*_{i_n}$. This is easily done by direct computation (in the sum on the right hand side, only one term may be nonzero). □

Exercise 9.12. *Let* $Y = \mathbb{R}^{\frac{n(n+1)}{2}+\frac{m(m-1)}{2}|mn}$ *be the real superspace of matrices*

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

(where A_{00} *is n by n and* A_{11} *is m by m) which are symmetric in the supersense, i.e.,* A_{00} *is symmetric,* A_{11} *is skew-symmetric, and* $A_{01}^T = A_{10}$*. Let* $Y_+ \subset Y$ *be the superdomain of those matrices for which* $A_{00} > 0$*. Let* dA *be a supervolume element on* Y*. Let* f *be a compactly supported smooth function on* Y_+*. Show that*

$$\int_{Y_+\times\mathbb{R}^{n|m}} f(A)e^{-x^TA_{00}x-2x^TA_{01}\xi-\xi^TA_{11}\xi}dAdx(d\xi)^{-1} =$$

$$= C\int_{Y_+} f(A)\mathrm{Ber}(A)^{-1/2}dA.$$

(C is a constant). What is C?

Exercise 9.13. *Prove the Amitsur-Levitzki identity: if* $X_1, ..., X_{2n}$ *are n by n matrices over a commutative ring, then*

$$\sum_{\sigma\in S_{2n}} (-1)^\sigma X_{\sigma(1)}...X_{\sigma(2n)} = 0.$$

Hint. *(a) Show that for any* n *by* n *matrix* X *with anticommuting entries,* $X^{2n} = 0$ *(namely, show that traces of* X^{2k} *vanish for all positive* k*, then use the Cayley-Hamilton theorem for* X^2*).*

(b) Apply this to $X = \sum_{i=1}^{2n} X_i \xi_i$*, where* ξ_i *are anticommuting variables.*