11. Free field theories in higher dimensions

11.1. **Minkowski and Euclidean space.** Now we pass from quantum mechanics to quantum field theory in dimensions $d \geq 1$. As we explained above, we have two main settings.

1. Minkowski space. Fields are functions on a spacetime $V = V_M$, which is a real inner product space of signature $(1, d-1)$. This is where physical processes actually "take place". The symmetry group of V, $G = SO(1, d-1)$, is called the *Lorentz group*; it is the group of transformations of spacetime in special relativity. Therefore, field theories in Minkowski space which are in an appropriate sense "compatible" with the action of G are called *relativistic*.

Recall some standard facts and definitions. The *light cone* in V is the cone described by the equation $|\mathbf{v}|^2 = 0$, where $|\mathbf{v}|^2 := (\mathbf{v}, \mathbf{v})$. Vectors belonging to the light cone are called *lightlike*. The light cone divides the space V into *spacelike vectors* $|\mathbf{v}|^2 < 0$ (outside the cone), and *timelike vectors* $|\mathbf{v}|^2 > 0$ (inside the cone). We will choose one of the two components of the interior of the cone and call it positive; it will be denoted by V_+. The opposite (negative) component is denoted by V_-. The group of $g \in SO(V) = SO(1, d-1)$ which preserve V_+ is denoted by $SO_+(1, d-1)$; it is the connected component of the identity of the group $SO(1, d-1)$ (which has two connected components).

Often (e.g. when doing Hamiltonian field theory) it is necessary to split V in an orthogonal direct sum $V = V_s \oplus \mathbb{R}$ of space and time. In this decomposition, the space V_s is required to be spacelike (i.e. negative definite), which implies that the time axis $\mathbb{R}$ has to be timelike (positive definite). Note that such a splitting is not unique, and that fixing it breaks the Lorentz symmetry $SO_+(1, d-1)$ down to the usual rotation group $SO(d-1)$.

To do explicit calculations, one further chooses Cartesian coordinates $x_1, ..., x_{d-1}$ on V_s and t on the time axis $\mathbb{R}$, so that $\mathbf{v} = (t, x_1, ..., x_{d-1})$. In these coordinates the inner product takes the form

$$|\mathbf{v}|^2 = c^2 t^2 - \sum_{j=1}^{d-1} x_j^2$$

where c is the speed of light. This explains the origin of the term "light cone" – it consists of worldlines of free photons (particles of light) traveling in space in some direction at speed c. To simplify notation, we will chose units of measurement so that $c = 1$.

2. Euclidean space. Fields are functions on a spacetime V_E, which is a positive definite inner product space. It plays an auxiliary role and has no direct physical meaning, although path integrals computed in this space are similar to expectation values in statistical mechanics.

The two settings are related by the "Wick rotation". Namely the Euclidean space V_E corresponding to the Minkowski space V_M is the real subspace in $(V_M)_{\mathbb{C}}$ consisting of vectors $(it, x_1, ..., x_{d-1})$, where t and x_j are real. In other words, to pass to the Euclidean space, one needs to make a change of variable $t \mapsto it$. Note that under this change, the standard metric on the Minkowski space, $dt^2 - \sum_j dx_j^2$ goes into a negative definite metric $-dt^2 - \sum_j dx_j^2$. However, the minus sign is traditionally dropped and one considers instead the positive metric $dt^2 + \sum_j dx_j^2$ on V_E.

11.2. **Free scalar boson.** Consider the theory of a free scalar bosonic field ϕ of mass m. The procedure of quantization of this theory in the Lagrangian setting is a straightforward generalization from the case of quantum mechanics. Namely, the Lagrangian for this theory in Minkowski space is

$$\mathcal{L} = \frac{1}{2}((d\phi)^2 - m^2\phi^2),$$

and the Euler-Lagrange equation is the *Klein-Gordon equation*

$$(\Box + m^2)\phi = 0,$$

where $\Box$ is the D'Alembertian (wave operator),

$$\Box := \frac{\partial^2}{\partial t^2} - \sum_j \frac{\partial^2}{\partial x_j^2}.$$

Thus to define the corresponding quantum theory, we should invert the operator $\Box + m^2$. This operator is essentially self-adjoint on compactly supported smooth functions and thus defines a self-adjoint operator, but as in the quantum mechanics case, it is not invertible – its spectrum is the whole $\mathbb{R}$, as can be easily seen by taking the Fourier transform. So as before, it is best to proceed using the Wick rotation.

After the Wick rotation (i.e. the transformation $t \mapsto it$), we arrive at the Euclidean Lagrangian

$$\mathcal{L}_E = \frac{1}{2}((d\phi)^2 + m^2\phi^2),$$

and the Euler-Lagrange equation is the Euclidean Klein-Gordon equation

$$(-\Delta + m^2)\phi = 0.$$

So to define the quantum theory, i.e. the path integral

$$\int \phi(x_1)...\phi(x_n)e^{-S(\phi)}D\phi$$

where $S = \int \mathcal{L}$, we now need to invert the self-adjoint operator $A = -\Delta + m^2$ (initially defined as an essentially self-adjoint operator on smooth compactly supported functions), whose spectrum is $[m^2, \infty)$, so it is invertible when $m > 0$. The operator A^{-1} is an integral operator whose Schwartz kernel is $G(x-y)$, where $G(x)$ is the Green's function, i.e. the fundamental solution of the Klein-Gordon equation:

$$-\Delta G + m^2 G = \delta.$$

To solve this equation, note that the solution is rotationally invariant. Therefore, outside of the origin, $G(x) = g(|x|)$, where g is a function on $(0, \infty)$ such that

$$-g'' - \frac{d-1}{r}g' + m^2 g = 0$$

(where the left hand side is the radial part of the operator A). This is a version of the Bessel equation. If $m > 0$, the two basic solutions are $r^{\frac{2-d}{2}} J_{\pm\frac{2-d}{2}}(imr)$, where J is the Bessel function. (Actually, these functions are elementary for odd d). Since we want G to decay at infinity (clustering property), we should pick the unique up to scaling linear combination which decays at infinity, namely,

$$g = Cr^{\frac{2-d}{2}}(J_{\frac{2-d}{2}}(imr) + i^d J_{-\frac{2-d}{2}}(imr)), \ d \neq 2. \tag{11.1}$$

For $d = 2$, this expression is zero, and one should instead take the limit of the right hand side divided by $d - 2$ as $d \to 2$. The normalizing constant can be found from the condition that $AG = \delta$.

Remark 11.1. It is easy to check that for $d = 1$ this function equals the familiar Green's function for quantum mechanics, $\frac{e^{-mr}}{2m}$.

If $m = 0$ (massless case), the basis of solutions is: $1, r$ for $d = 1$, $1, \log r$ for $d = 2$, and $1, r^{2-d}$ for $d > 2$. Thus, if $d \leq 2$, we don't have a decaying solution and thus the corresponding quantum theory will be deficient: it will not satisfy the clustering property. On the other hand, for $d > 2$ we have a unique up to scaling decaying solution $g = Cr^{1-d}$. The normalizing constant is found as in the massive case.

The higher correlation functions are found from the 2-point function via the Wick formula, as usual.

We should now note a fundamental difference between quantum mechanics and quantum field theory in $d > 1$ dimensions. This difference comes from the fact that while for $d = 1$, the Green's function $G(x)$ is continuous at $x = 0$, for $d > 1$ it is singular at $x = 0$. Namely, $G(x)$ behaves like $C|x|^{2-d}$ as $x \to 0$ for $d > 2$, and as $C \log |x|$ as $d = 2$.

Thus for $d > 1$, unlike the case $d = 1$, the path integral

$$\int \phi(x_1)...\phi(x_n)e^{-S(\phi)}D\phi$$

(as defined above) makes sense only if $x_i \neq x_j$. In other words, this path integral should be regarded not as a function but rather as a distribution. Luckily, there is a canonical way to do it, since the Green's function $G(x)$ is locally L^1.

Now we can Wick rotate this theory back into the Minkowski space. It is clear that the Green's function will then turn into

$$G_M(x) = g(\sqrt{-|x|^2 - i\varepsilon}),$$

which involves Bessel functions of both real and imaginary argument (depending on whether x is timelike or spacelike) and has a singularity on the light cone $|x|^2 = 0$. In particular, it is easy to check that $G_M(x)$ is real-valued for spacelike x, while for timelike x it is not. The function $G_M(x)$ satisfies the equation

$$(\Box + m^2)G_M = i\delta.$$

The higher correlation functions, as before, are determined from this by the Wick formula.

Actually, it is more convenient to describe this theory "in momentum space", where the Green's function can be written more explicitly. Namely, the Fourier transform $\widehat{G}(p)$ of the distribution $G(x)$ is a solution of the equation

$$p^2\widehat{G} + m^2\widehat{G} = 1,$$

obtained by Fourier transforming the differential equation for G. Thus,

$$\widehat{G}(p) = \frac{1}{p^2 + m^2},$$

as in the quantum mechanics case. Therefore, like in quantum mechanics, the Wick rotation produces the distribution

$$\widehat{G}_M(p) = \frac{i}{p^2 - m^2 + i\varepsilon},$$

which is the Fourier transform of $G_M(x)$.

11.3. **Spinors.** To consider field theory for fermions, we must generalize to the case of $d > 1$ the basic fermionic Lagrangian $\frac{1}{2}\psi\frac{d\psi}{dt}$. To do this, we must replace $\frac{d}{dt}$ by some differential operator on V. This operator should be of first order, since in fermionic quantum mechanics it was important that the equations of motion are first order equations. Clearly, it is impossible to define such an operator so that the Lagrangian is $SO_+(V)$-invariant, if ψ is a scalar-valued (odd) function on V. Thus, a fermionic field in field theory of dimension $d > 1$ cannot be scalar-valued, but rather must take values in a real representation S of $SO_+(V)$, such that there exists a nonzero intertwining operator $V \to \mathrm{Sym}^2 S^*$. This property is satisfied by *spinor representations*. They are indeed basic in fermionic field theory, and we will now briefly discuss them (for more detail see "Spinors" by P.Deligne, in "QFT and string theory: a course for mathematicians").

First consider the complex case. Let V be a complex inner product space of dimension $d > 1$. Let $\mathrm{Cl}(V)$ be the Clifford algebra of V, defined by the relation $\xi\eta + \eta\xi = 2(\xi, \eta)$, $\xi, \eta \in V$. As we discussed, for even d it is simple and has a unique irreducible representation S of dimension $2^{\frac{d}{2}}$, while for odd d it has two such representations S', S'' of dimension $2^{\frac{d-1}{2}}$. It is easy to show that the space $\mathrm{Cl}_2(V)$ of quadratic elements of $\mathrm{Cl}(V)$ (i.e. the subspace spanned elements of the form $\xi\eta - \eta\xi$, $\xi, \eta \in V$) is closed under bracket, and constitutes the Lie algebra $\mathfrak{o}(V)$. Thus $\mathfrak{o}(V)$ acts on S (respectively, S', S''). This action does not integrate to an action of $SO(V)$, but integrates to an action of its double cover $\mathrm{Spin}(V)$.

If d is even, the representation S of $\mathrm{Spin}(V)$ is not irreducible. Namely, recall that S is the exterior algebra of a Lagrangian subspace of V. Thus it splits in a direct sum $S = S_+ \oplus S_-$ (odd and even elements). The subspaces S_+, S_- are subrepresentations of S, which are irreducible. They are called the *half-spinor representations*. The half-spinor representations are interchanged by the adjoint action of $O(V)$ on $\mathrm{Spin}(V)$ ($SO(V)$ clearly acts trivially, so this is, in fact, and action of $O(V)/SO(V) = \mathbb{Z}/2$ on the set of irreducible representations of $SO(V)$). Note that in contrast, for odd d we have $O(V) = SO(V) \times \mathbb{Z}/2$, so the $\mathbb{Z}/2$ acts on representations of $\mathrm{Spin}(V)$ trivially.

If d is odd, the representations S' and S'' of $\mathrm{Spin}(V)$ are irreducible and isomorphic. Any of them will be denoted by S and called the *spinor representation*. Thus, we have the spinor representation S for both odd and even d, but for even d it is reducible.

An important structure attached to the spinor representation S is the intertwining operator $\Gamma : V \to \mathrm{End} S$ called *Clifford multiplication*, given by the action of $V \subset \mathrm{Cl}(V)$ in S, which we already encountered above. This intertwiner allows us to define the *Dirac operator*

$$\mathbf{D} = \sum_i \Gamma_i \frac{\partial}{\partial x_i} \tag{11.2}$$

where x_i are coordinates on V associated to an orthornormal basis e_i, and $\Gamma_i = \Gamma(e_i)$. This operator acts on functions from V to S, and $\mathbf{D}^2 = \Delta$, so $\mathbf{D}$ is a square root of the Laplacian. The matrices Γ_i are called Γ-*matrices*.

Note that for even d, one has $\Gamma(v) : S_\pm \to S_\mp$, so $\mathbf{D}$ acts from functions with values in $S_\pm$ to functions with values in $S_\mp$.

By a *polyspinor representation* of $\mathrm{Spin}(V)$ we will mean any linear combination of S_+, S_- for even d, and any multiple of S for odd d. For even d and a polyspinor representation $Y = Y_+ \otimes S_+ \oplus Y_- \otimes S_-$ (i.e., $Y_\pm = \mathrm{Hom}(S_\pm, Y)$) where Y_+, Y_- are vector spaces, set $Y' := Y_+ \otimes S_- \oplus Y_- \otimes S_+$, while for odd d and $Y = Y_0 \otimes S$ we set $Y' := Y$; thus $Y \mapsto Y'$ is an endofunctor on the category of polyspinor representations. Then for every polyspinor representation Y and $v \in V$ we have the Clifford multiplication operator $\Gamma(v) : Y \to Y'$.

Now assume that V is a real inner product space with Minkowski metric. In this case we can define the group $\mathrm{Spin}_+(V)$ to be the preimage of $SO_+(V)$ under the map $\mathrm{Spin}(V_\mathbb{C}) \to SO(V_\mathbb{C})$. It is a double cover of $SO_+(V)$ (if $d = 2$, this double cover is disconnenced and actually a direct product by $\mathbb{Z}/2$).

By a *real polyspinor representation* of $\mathrm{Spin}_+(V)$ we will mean a real representation Y of this group such that $Y_\mathbb{C}$ is a polyspinor representation of $\mathrm{Spin}(V_\mathbb{C})$.

Remark 11.2. Note that in all dimensions except $d = 2$, the group $\mathrm{Spin}(d)$ is the universal cover of $SO(d)$, which means that spins of all particles are either integers or half-integers. On the other hand, the universal cover of $SO(2)$ is not $\mathrm{Spin}(2)$, but rather $\mathbb{R}$. This creates in two dimensions a possibility of particles whose spin is any positive real number. Such particles are called *anyons* (particles of any spin), and we will see how they appear in 2-dimensional conformal field theory.

11.4. **Fermionic Lagrangians.** Now let us consider Lagrangians for a spinor field ψ with values in a polyspinor representation Y. Note that in even dimensions such fields are split into fields valued in S_+ and S_-, respectively. Such spinors are called *chiral*.

As the Lagrangian is supposed to be real in the Minkowski setting, we will require in that case that Y be real. First of all, let us see what we need in order to write the "kinetic term" $(\psi, \mathbf{D}\psi)$. Clearly, to define such a term (so that the corresponding term in the action does not reduce to zero via integration by parts), we need an invariant non-degenerate pairing $(,)$ between Y and Y' (i.e., an isomorphism of representations $Y' \cong Y^*$) such that for any $v \in V$, the bilinear form $(x, \Gamma(v)y)$ on Y is symmetric.

Let us find for which Y this is possible (for complex V). The behavior of Spin groups depends on d modulo 8 (*real Bott periodicity*). Thus we will list the answers labeling them by d mod 8 (they are easily extracted from the tables given in Deligne's text). First we summarize properties of spin representations.

0. $S_\pm$ orthogonal.
1. S orthogonal, $S \otimes S \to V$ symmetric.
2. $S_+^* = S_-$, $S_\pm \otimes S_\pm \to V$ symmetric.
3. S symplectic, $S \otimes S \to V$ symmetric.
4. $S_\pm$ symplectic.
5. S symplectic, $S \otimes S \to V$ antisymmetric.
6. $S_+ = S_-^*$, $S_\pm \otimes S_\pm \to V$ antisymmetric.
7. S orthogonal, $S \otimes S \to V$ antisymmetric.

Thus the possibilities for the kinetic term are:

0. $n(S_+ \oplus S_-)$; $(,)$ gives a perfect pairing between Y_+ and Y_-.
1. nS; $(,)$ gives a symmetric inner product on Y_0.
2. $nS_+ \oplus kS_-$; $(,)$ gives symmetric inner products on $Y_\pm$.
3. nS; $(,)$ gives a symmetric inner product on Y_0.
4. $n(S_+ \oplus S_-)$; $(,)$ gives a perfect pairing between Y_+ and Y_-.
5. $2nS$; $(,)$ gives a skew-symmetric inner product on Y_0.
6. $2nS_+ \oplus 2kS_-$; $(,)$ gives skew-symmetric inner products on $Y_\pm$.
7. $2nS$; $(,)$ gives a skew-symmetric inner product on Y_0.

Let us now find when we can also add a mass term. Recall that the mass term has the form $(\psi, M\psi)$, so it corresponds to an invariant skew-symmetric operator $M : Y \to Y^* \cong Y'$ (note that by definition, Γ_i commute with M). Let us list those Y from the above list for which such a non-degenerate operator exists.

0. $2n(S_+ \oplus S_-)$; $M_\pm : Y_\pm \to Y_\mp$ are skew-symmetric under $(,)$.
1. $2nS$; $M : Y_0 \to Y_0$ is skew-symmetric under $(,)$.
2. $n(S_+ \oplus S_-)$; $M_\pm : Y_\pm \cong Y_\mp$ satisfy $M_+^* = -M_-$ under $(,)$.
3. nS; $M : Y_0 \to Y_0$ is symmetric under $(,)$.
4. $n(S_+ \oplus S_-)$; $M_\pm : Y_\pm \to Y_\mp$ are symmetric under $(,)$.
5. $2nS$; $M : Y_0 \to Y_0$ is symmetric under $(,)$.
6. $2n(S_+ \oplus S_-)$; $M_\pm : Y_\pm \cong Y_\mp$ satisfy $M_+^* = -M_-$ under $(,)$.
7. $2nS$; $M : Y_0 \to Y_0$ is skew-symmetric under $(,)$.

To pass to the real Minkowski space (in both massless and massive case), one should put the additional requirement that Y should be a real representation.

We note that upon Wick rotation to Minkowski space, it may turn out that a real spinor representation Y will turn into a complex representation which has no real structure. Namely, this happens for massless spinors that take values in $S_\pm$ if $d = 2$ mod 8. These representations have a real structure for Minkowskian V (i.e. for $\mathrm{Spin}_+(1, d-1)$), but no real structure for Euclidean V (i.e. for $\mathrm{Spin}(d)$). This is quite obvious, for example, when $d = 2$ (check!).

Remark 11.3. One may think that this causes a problem in quantum field theory, where we would be puzzled what to integrate over – real or complex space. However, the problem in fact does not arise, since we have to integrate over fermions, and integration over fermions (say, in the finite dimensional case) is purely algebraic and does not make a distinction between real and complex.

11.5. **Free fermions.** Let us now consider a free theory for a spinor field $\psi : V \to \Pi Y$, where Y is a polyspinor representation, defined by a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\psi, (\mathbf{D} - M)\psi),$$

where M is allowed to be degenerate (we assume that Y is such that this expression makes sense). The equation of motion in Minkowski space is

$$\mathbf{D}\psi = M\psi.$$

Thus, to define the corresponding quantum theory, we need to invert the operator $\mathbf{D} - M$. As usual, this cannot be done because of a singularity, and it is best to use the Wick rotation.

The Wick rotation produces the Euclidean Lagrangian

$$\mathcal{L} = \frac{1}{2}(\psi, (\mathbf{D}_E + M)\psi)$$

(note that the i in the kinetic term is hidden in the definition of the Euclidean Dirac operator). We invert $\mathbf{D}_E + M$ to obtain the Euclidean Green's function. To do this, it is convenient to go to momentum space, i.e. perform a Fourier transform. Namely, after Fourier transform $\mathbf{D}_E$ turns into the operator $i\mathbf{p}$, where $\mathbf{p} = \sum_j p_j \Gamma_j$, and p_j are the operators of multiplication by the momentum coordinates p_j. Thus, the Green's function (i.e. the 2-point function) $G(x) \in \mathrm{Hom}(Y^*, Y)$ is the Fourier transform of the matrix-valued function $\frac{1}{i\mathbf{p}+M}$.

In the Euclidean case the group $\mathrm{Spin}(V)$ is compact and the spinor representations carry natural positive invariant Hermitian forms. So in this case without loss of generality we may consider polyspinor representations equipped with such positive forms, and on every polyspinor representation such a form is unique up to isomorphism. Let

$$M^\dagger : Y^* \to Y$$

be the Hermitian adjoint operator to M. Then the reality condition is that M is Hermitian: $M^\dagger = M$. Thus

$$(-i\mathbf{p} + M)(i\mathbf{p} + M) = p^2 + M^2$$

so that

$$\widehat{G}(p) = (p^2 + M^2)^{-1}(-i\mathbf{p} + M).$$

This shows that $G(x)$ is expressed through the Green's function in the bosonic case by differentiations (how?). After Wick rotation back to the Minkowski space, we get

$$\widehat{G}_M(p) = (p^2 - M^2 + i\varepsilon)^{-1}(\mathbf{p} + iM).$$

Finally, the higher correlation functions, as usual, are found from the Wick formula.

11.6. **Hamiltonian formalism of classical field theory.** Let us now develop the hamiltonian approach to QFT, extending the hamiltonian formalism of quantum mechanics. We start with classical field theory, extending the hamiltonian formalism of classical mechanics. As in the Lagrangian setting, this can be done by formalizing the idea that field theory is mechanics of a continuum of particles occupying each point of the space $\mathbb{R}^{d-1}$.

Namely, consider a free scalar bosonic field $\phi(x)$ on a Minkowski space $\mathbb{R}^d$. As we have discussed, its Lagrangian is $\mathcal{L} = \frac{1}{2}((d\phi)^2 - m^2\phi^2)$ and the equation of motion is the Klein-Gordon equation

$$\phi_{tt} - \Delta_s\phi + m^2\phi = 0,$$

where Δ_s is the spacial Laplacian. This is a second order equation with respect to t, so the initial value problem for this equation has the form

$$\phi(0, x) = q(x), \ \phi_t(0, x) = p(x)$$

(there is a standard explicit formula for solution of this problem, expressing it via the fundamental solution of the Klein-Gordon equation). Thus it is natural to introduce the phase space

$$Y := T^*C_0^\infty(\mathbb{R}^{d-1}) := C_0^\infty(\mathbb{R}^{d-1}) \oplus C_0^\infty(\mathbb{R}^{d-1})$$

of pairs (q, p) of smooth functions with compact support, on which the dynamics of the Klein-Gordon equation takes place (note that the space $C_0^\infty(\mathbb{R}^{d-1})$ is invariant under this dynamics since the speed of wave propagation is finite, namely equals 1). Note that the phase space is an infinite dimensional symplectic space with constant symplectic form

$$\omega((q_1,p_1),(q_2,p_2)) = \int_{\mathbb{R}^{d-1}} (p_1(x)q_2(x) - p_2(x)q_1(x))dx.$$

Also for any point $x \in \mathbb{R}^{d-1}$ we have the local linear functionals

$$(q,p) \mapsto q(x), \ (q,p) \mapsto p(x)$$

which we will denote by $\phi(x)$ and $\phi_t(x)$, respectively. From these functionals we can make other linear functionals: for example, given $\rho \in C_0^\infty(\mathbb{R}^{d-1})$, we can define the functionals

$$\phi(\rho)(q,p) := \int_{\mathbb{R}^{d-1}} q(x)\rho(x)dx, \ \phi_t(\rho)(q,p) := \int_{\mathbb{R}^{d-1}} p(x)\rho(x)dx.$$

The Poisson bracket between such functionals can be computed by the formulas

$$\{\phi(\rho_1),\phi(\rho_2)\} = 0, \ \{\phi_t(\rho_1),\phi_t(\rho_2)\} = 0,$$
$$\{\phi(\rho_1),\phi_t(\rho_2)\} = \int_{\mathbb{R}^{d-1}} \rho_1(x)\rho_2(x)dx.$$

This can be written as a *field-theoretic Poisson bracket*:

$$\{\phi(x),\phi(y)\} = 0, \ \{\phi_t(x),\phi_t(y)\} = 0, \ \{\phi(x),\phi_t(y)\} = \delta(x-y);$$

then the previous formulas can be recovered by integrating both sides against $\rho_1(x)\rho_2(y)$. In other words, the linear local functionals $\phi(x)$ and $\phi_t(x)$ should be thought of not as smooth functions on Y depending on a point $x \in \mathbb{R}^{d-1}$ but rather as distributions on $\mathbb{R}^{d-1}$ with values in smooth functions on Y.

Similarly, one may consider non-linear polynomial local functionals, given by differential polynomials $P(\phi,\phi_t)$ evaluated at a point x, such as $\phi^n, \phi_t^2, (d_s\phi)^2, \phi^2(d_s\phi)^2$ (where d_s is the spatial differential), etc., and even non-polynomial ones depending on finitely many derivatives of ϕ, such as $e^\phi(d_s\phi)^2, \cos\phi$, and so on. They are called local because they depend only on the derivatives of ϕ at a single point x. Each of them is a distribution on $\mathbb{R}^{d-1}$ with values in smooth functions on Y, and can be applied to any density $\rho(x)$ to produce a smooth function on Y. Poisson brackets of such functionals are computed using the chain rule, the Leibniz rule, and the fact that taking Poisson brackets commutes with differentiation by x. So given two local functionals P and Q, we obtain

$$\{P(\phi)(x),Q(\phi)(y)\} = \sum_\alpha \{P,Q\}_\alpha(\phi)(x)\partial_x^\alpha \delta(x-y)$$

for some local functionals $\{P,Q\}_\alpha$, where ∂^α are monomials in the derivatives. For example, for $d=2$

$$\{\phi_{tx}(u)\phi_t(u), \tfrac{1}{3}\phi^3(v)\} =$$

$$-(\phi_{tx}(u)\phi^2(u) + 2\phi_{tx}(u)\phi(u)\phi_x(u))\delta(u-v) - \phi_t(u)\phi^2(u)\delta'(u-v).$$

This Poisson bracket can of course be extended to products of local functionals at different points using the Leibniz rule.

The Hamiltonian of the theory is then given by integrating a local functional against the constant density:

$$H(\phi) = \frac{1}{2}\int_{\mathbb{R}^{d-1}} (\phi_t^2 + (d_s\phi)^2 + m^2\phi^2)dx.$$

Namely, it is determined (up to a constant) by the condition that the Hamilton equation

$$F_t = \{F, H\}$$

for local functionals ϕ, ϕ_t is equivalent to the Klein-Gordon equation.

The Hamiltonian dynamics allows us to define the local functionals not just at a point $x \in \mathbb{R}^{d-1}$ but actually at any point $(t,x) \in \mathbb{R}^d$. When we do, by definition we get $\phi_t(t,x) = \frac{d}{dt}\phi(t,x)$ and the local functional $\phi(t,x)$ becomes a solution of the Klein-Gordon equation:

$$\phi_{tt} - \Delta_s\phi + m^2\phi = 0.$$

This can be used to compute the Poisson brackets: for example, we see that

$$\{\phi(t_1,x_1), \phi(t_2,x_2)\} = \mathbf{G}(t_2-t_1, x_2-x_1)$$

where $\mathbf{G}(t,x)$ solves the Klein-Gordon equation with initial conditions

$$\mathbf{G}(0,x) = 0,\ \ \mathbf{G}_t(0,x) = \delta(x).$$

To find it, take the Fourier transform. Then we get a distribution $\widehat{\mathbf{G}}$ supported on the two-sheeted hyperboloid X_m given by the equation $E^2 = p^2 + m^2$, of the form

$$\widehat{\mathbf{G}}(E,p) = f_+(p)\delta_{X_m^+} + f_-(p)\delta_{X_m^-},$$

where $X_m^\pm$ are the sheets of X_m. Moreover, the initial conditions give (up to appropriate normalization)

$$\int_{\mathbb{R}} \widehat{\mathbf{G}}(E,p)dE = 0,\ \int_{\mathbb{R}} \widehat{\mathbf{G}}(E,p)EdE = 1,$$

which yields

$$f_+(p) + f_-(p) = 0,\ \sqrt{p^2+m^2}(f_+(p) - f_-(p)) = 1.$$

Thus $f_+ = -f_- = \frac{1}{2\sqrt{p^2+m^2}}$ and we have

$$\widehat{\mathbf{G}}(E,p) = \frac{1}{2\sqrt{p^2+m^2}}(\delta_{X_m^+} - \delta_{X_m^-}).$$

Now $\mathbf{G}$ can be found by taking the inverse Fourier transform (it expresses via the Bessel functions).

Note that since the speed of wave propagation is 1, this distribution $\mathbf{G}$ is supported on the solid light cone, so $\{\phi(t_1,x_1),\phi(t_2,x_2)\} = 0$ if the points (t_1,x_1) and (t_2,x_2) are *spacelike separated*, meaning that the vector (t_1-t_2, x_1-x_2) is spacelike. This property is called *space locality*, a mathematical expression of *causality* in special relativity.

Remark 11.4. A part of this analysis extends straightforwardly to the case of non-free theories, for example the ϕ^4*-theory*, having the Lagrangian

$$\mathcal{L} = \frac{1}{2}((d\phi)^2 - m^2\phi^2) - \frac{g}{4}\phi^4.$$

In this case the Klein-Gordon equation is replaced by its non-linear deformation

$$\phi_{tt} - \Delta_s\phi + m^2\phi + g\phi^3 = 0,$$

so there is a nontrivial issue of existence of solutions of the initial value problem for this non-linear PDE. However, this issue is irrelevant if we just want to consider Poisson brackets of local functionals on $\mathbb{R}^{d-1}$ or its formal neighborhood, since then the computations are purely formal (algebraic).

An important fact is that this structure is invariant under the *Poincaré group* $\mathbf{P} := SO_+(V) \ltimes V$ generated by Minkowski rotations and translations, where $V = \mathbb{R}^d$ is the spacetime (the semidirect product of the *Lorentz group* $SO_+(V)$ and the group of translations V). This follows from the fact that the Lagrangian of the theory is relativistically invariant. Namely, for $g \in \mathbf{P}$ given by

$$g(t,x) = (at + bx + c, \alpha t + \beta x + \gamma)$$

we have

$$(\phi g)(x)(q,p) = \phi(bx + c, \beta x + \gamma)$$

and

$$(\phi_t g)(x)(q,p) = (\partial_\alpha \phi)(bx + c, \beta x + \gamma) + a\phi_t(bx + c, \beta x + \gamma).$$

where $\phi(t,x)$ is the solution of the Klein-Gordon equation with initial conditions $(q(x), p(x))$.

In particular, note that the *Galileo subgroup* $SO(\mathbb{R}^{d-1}) \ltimes \mathbb{R}^{d-1}$ acts by manifest geometric symmetries, while time translations act by the Hamiltonian flow.

Finally, note that this discussion extends in a straighforward way to theories including fermions. In this case, as in fermionic classical mechanics, we get a field theoretic *super*-Poisson bracket on classical fields, which is symmetric rather than skew-symmetric if both fields are odd. Also, since odd fields take values in polyspinor representations, the Poincaré group should be replaced by its double cover $\widetilde{\mathbf{P}} := \mathrm{Spin}_+(V) \ltimes V$. We leave the details to the reader.

11.7. **Hamiltonian formalism of QFT: the Wightman axioms.** To quantize this picture, we need to define a Hilbert space $\mathcal{H}$ and lift classical observables (local functionals and their integrals) to (densely defined) operators on $\mathcal{H}$, notably lift the classical hamiltonian H to a quantum hamiltonian $\widehat{H}$ depending on the Planck constant $\hbar$ which should be a self-adjoint (in general, unbounded) operator on $\mathcal{H}$. Moreover, this should be done in such a way that commutators vanish at $\hbar = 0$ and in first order in $\hbar$ recover the Poisson bracket. We should also have a unitary representation of the double cover $\widetilde{\mathbf{P}}$ of the Poincaré group on the space $\mathcal{H}$ such that the 1-parameter subgroup of time translations acts by the quantum dynamics 1-parameter group $e^{-it\widehat{H}}$. This generalization of Hamiltonian quantum mechanics can be accomplished by means of so called *Wightman axioms*, which we now describe.

First of all, for the quantum theory to have good properties, we want the energy to be bounded below. Thus we introduce the following definition. Let us fix an orthogonal decomposition $V = \mathbb{R} \oplus V_s$ into space and time and consider the self-adjoint operator

$$\widehat{H}_\pi := i\tfrac{d}{dt}|_{t=0}\pi(t,0).$$

Definition 11.5. A unitary representation $\pi : \widetilde{\mathbf{P}} \to \mathrm{Aut}\mathcal{H}$ is said to be *positive energy* if the spectrum of $\widehat{H}_\pi$ is bounded below.

Note that every unitary representation π of V has a spectrum $\sigma(\pi)$, which is a closed subset of $V^* \cong V$; namely, $\sigma(\pi)$ is the set of characters of V that occur (discretely or continuously) in π (i.e., the smallest set containing the support of the Fourier transform of the distribution $\langle w_1, \pi(v)w_2 \rangle$, $v \in V$, for any $w_1, w_2 \in \mathcal{H}$).

Lemma 11.6. *Suppose* $\dim V \geq 2$. *Then* π *is positive energy if and only if* $\sigma(\pi)$ *is contained in the positive part of the solid light cone,* $\overline{V}_+$.

Proof. By definition, π is of positive energy iff the orthogonal projection of $\sigma(\pi)$ onto the dual of the time axis is bounded below. Since $\sigma(\pi)$ is invariant under $SO_+(V)$, this implies the statement (an $SO_+(V)$-orbit on V has bounded below projection iff it is contained in $\overline{V}_+$). $\square$

Note that this is false for $d = 1$ (quantum mechanics), where the hamiltonian can be shifted by a constant without any effect on the theory. But the latter is longer so in quantum field theory on a Minkowski space of dimension > 1.

We are now ready to give Wightman's definition of a QFT. Let $\mathcal{S} = \mathcal{S}(V)$ be the Schwartz space of V.

Definition 11.7. A *Wightman QFT* on a Minkowski space V entails the following data:

1. A finite dimensional real super-representation $R = R_0 \oplus R_1$ of $\mathrm{Spin}_+(V)$ (the *field space*).

2. A super Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ carrying a positive energy unitary representation $\pi : \widetilde{\mathbf{P}} \to \mathrm{Aut}\mathcal{H}$ of the double cover of the Poincaré group, $\widetilde{\mathbf{P}} = \mathrm{Spin}_+(V) \ltimes V$.

3. A dense $\widetilde{\mathbf{P}}$-stable subspace $\mathcal{D} \subset \mathcal{H}$.

4. A $\widetilde{\mathbf{P}}$-invariant unit vector $\Omega \in \mathcal{D}$ called the *vacuum vector*.

5. A $\widetilde{\mathbf{P}}$-invariant even linear map: $\mathcal{S} \otimes R^* \to \mathrm{End}\mathcal{D}$ called the *field map*.

This data is subject to the following axioms.

A1. If f is real then $\phi(f)$ is Hermitian symmetric (in the supersense).

A2. ϕ is weakly continuous, i.e. for every $w_1, w_2 \in \mathcal{D}$, the functional $\mathcal{S} \otimes R^* \to \mathbb{C}$ defined by $f \mapsto \langle w_1, \phi(f) w_2 \rangle$ is continuous.

A3. $\mathcal{D}$ is spanned (algebraically) by vectors $\phi(f_1)...\phi(f_n)\Omega$.

A4. *Space locality*: If f_1, f_2 have spacelike separated supports, i.e., for any $v_1 \in \mathrm{supp} f_1$, $v_2 \in \mathrm{supp} f_2$ we have $|v_1 - v_2|^2 < 0$, then

$$[\phi(f_1), \phi(f_2)] = 0$$

(with commutator understood in the supersense).

In addition, if $\mathcal{H}^{\widetilde{\mathbf{P}}} = \mathbb{C}\Omega$, one says that we have a Wightman QFT with a *unique vacuum*.

We will also always assume that our QFT is *nondegenerate*, i.e., for every irreducible subrepresentation $E \subset R_j^*$, $j = 0, 1$, one has $\phi|_{\mathcal{S}\otimes E} \neq 0$; otherwise we can simply remove this subrepresentation without any effect on the theory.

A fundamental fact about Wightman QFT is the following theorem, which we will not prove here. Let ζ be the generator of the kernel of the map $\mathrm{Spin}_+(V) \to \mathrm{SO}_+(V)$, so $\zeta^2 = 1$.

Theorem 11.8. *(The spin-statistics theorem) If $E \subset R_j^*$ is a subrepresentation then $\zeta|_E = (-1)^j$.*

In other words, there is a relationship between the *spin* (mod integers) of a quantum field (essentially, the eigenvalue of ζ) and its *statistics*, i.e., whether it is bosonic (even) or fermionic (odd). Namely, the theorem says that all bosonic fields must have $\zeta = 1$ (integer spin) and all fermionic fields must have $\zeta = -1$ (half-integer spin).

Remark 11.9. We will see that the theory of free bosons and fermions can be naturally formulated as a Wightman QFT. Moreover, this is also the case for a number of non-free theories, which is the subject of a difficult area of mathematical physics called *constructive field theory.* Still, most theories that physicists really care about are either not known to be Wightman QFT, or simply fail to be ones for various reasons (perturbative theories, low energy effective theories, non-unitary theories, Euclidean theories, theories living on compact manifolds, etc.) Thus we will view Wightman axioms just as one (somewhat limited) rigorous model for our mathematical understanding of QFT.

11.8. **Wightman functions.**

Proposition 11.10. *In a Wightman QFT on a Minkowski space V, for every $n \geq 1$ there exists a unique tempered distribution W_n on V^n valued in $R^{*\otimes n}$ such that*

$$W_n(f_1 \boxtimes ... \boxtimes f_n) = \langle \Omega, \phi(f_1)...\phi(f_n)\Omega \rangle.$$

We leave the proof of this proposition as an exercise.

We will therefore think of W_n as a (generalized) function on $V^{\otimes n}$ valued in $R^{*\otimes n}$, denoted $W_n(x_1, ..., x_n)$, so that

$$W_n(f_1 \boxtimes ... \boxtimes f_n) = \int_{V^n} W_n(x_1, ..., x_n) f_1(x_1)...f_n(x_n) dx_1...dx_n$$

where the product on the right hand side involves contraction of corresponding copies of R and R^*. Thus, given $u_1, ..., u_n \in R$, we have the scalar-valued distribution

$$W_n^{u_1,...,u_n}(x_1, ..., x_n) := (W_n(x_1, ..., x_n), u_1 \otimes ... \otimes u_n).$$

In other words, we may define an operator-valued distribution $\phi(x)$ such that

$$\phi(f) = \int_V \phi(x) f(x) dx;$$

then

$$W_n(x_1, ..., x_n) = \langle \Omega, \phi(x_1)...\phi(x_n)\Omega \rangle.$$

Definition 11.11. The generalized functions $W_n(x_1, ..., x_n)$ are called the *Wightman (correlation) functions* of the Wightman QFT.

Note that Wightman functions completely determine the Wightman QFT as follows. Let $\widetilde{\mathcal{D}} := T(\mathcal{S} \otimes R)$ (the tensor algebra), so it is spanned by elements $f_1 \otimes f_2 \otimes ... \otimes f_n$, $f_i \in \mathcal{S} \otimes R$. Define the inner product on $\widetilde{\mathcal{D}}$ by

$$\langle f_1 \otimes ... \otimes f_n, g_1 \otimes ... \otimes g_m \rangle := (-1)^{\sum_{i<j} p(f_i)p(f_j)} W_{n+m}(\overline{f}_n \boxtimes ... \boxtimes \overline{f}_1 \boxtimes g_1 \boxtimes ... \boxtimes g_m).$$

It is easy to see that this inner product is well defined, and

$$\langle f_1 \otimes ... \otimes f_n, g_1 \otimes ... \otimes g_m \rangle = \langle \phi(f_1)...\phi(f_n)\Omega, \phi(g_1)...\phi(g_m)\Omega \rangle$$

(where f_i are purely odd or purely even). Thus the inner product $\langle\, , \rangle$ on $\widetilde{\mathcal{D}}$ is nonnegative definite, the Hilbert space $\mathcal{H}$ can be recovered as the completion of $\widetilde{\mathcal{D}}$ with respect to $\langle\, , \rangle$, and $\mathcal{D}$ is the image of $\widetilde{\mathcal{D}}$ in $\mathcal{H}$ (note that the map $\widetilde{\mathcal{D}} \to \mathcal{H}$ need not be injective). Moreover, the vector Ω is the image of $1 \in \widetilde{\mathcal{D}}$ in $\mathcal{D}$, and the representation π is obtained by extending the action of $\widetilde{\mathbf{P}}$ on $\mathcal{D}$ (which descends from $\widetilde{\mathcal{D}}$) by continuity.

So we can ask: what conditions should Wightman functions satisfy to define a Wightman QFT? Let us list some necessary conditions, which follow from the above discussion. To this end, denote by

$$W : T(\mathcal{S} \otimes R) \to \mathbb{C}$$

the natural liner map and by $* : T(\mathcal{S} \otimes R) \to T(\mathcal{S} \otimes R)$ the antilinear map given by $(f_1 \otimes ... \otimes f_n)^* = (-1)^{\sum_{i<j} p(f_i)p(f_j)} \overline{f}_n \otimes ... \otimes \overline{f}_1$.

Proposition 11.12. *The Wightman functions W_n of a Wightman QFT satisfy the following properties.*

1. W_n are $\widetilde{\mathbf{P}}$-invariant.

2. Positive energy: the Fourier transform of W_n is supported on the set of $(p_1, ..., p_n) \in V^n$ such that $\sum_i p_i = 0$ and $p_{i+1} - p_i \in \overline{V}_+$.

3. $W_n(f^*) = \overline{W_n(f)}$.

4. Space locality:

$$W_n^{u_1,...,u_n}(x_1, ..., x_i, x_{i+1}, ..., x_n) = (-1)^{p(u_i)p(u_{i+1})} W_n(x_1, ..., x_{i+1}, x_i, ..., x_n)$$

if $|x_i - x_{i+1}|^2 < 0$.

5. Positivity: $W(f^* \otimes f) \geq 0$ for any $f \in T(\mathcal{S} \otimes R)$.

Proof. (1) follows from the invariance of the vacuum vector and the field map. (3) follows from the fact that for real f, $\phi(f)$ is hermitian symmetric. (4) follows from the space locality axiom. (5) follows from positivity of the inner product on $\mathcal{H}$. So it remains to prove (2). Let us do so for $n = 2$, the general proof is similar.

By translation invariance we have

$$W_2(v_1, v_2) = \mathbb{W}(v)$$

where $v = v_2 - v_1$. Thus our job is to show that the Fourier transform of $\mathbb{W}$ is supported on $\overline{V}_+$. We have

$$\mathbb{W}(v) = \langle\, \Omega, \phi(0)\phi(v)\Omega \,\rangle =$$
$$= \langle\, \Omega, \phi(0)\pi(v)\phi(0)\pi(-v)\Omega \,\rangle = \langle\, \phi(0)\Omega, \pi(v)\phi(0)\Omega \,\rangle.$$

So the statement follows from the fact that every character of V which occurs in $\mathcal{H}$ belongs to $\overline{V}_+$. □

In fact, it turns out that these necessary conditions are also sufficient, and we have the following theorem, which can be proved by following the above reconstruction procedure (but we will not give a proof):

Theorem 11.13. *If a collection of distributions W_n satisfies conditions (1)-(5) of Proposition 11.12 then they define a Wightman QFT.*

Remark 11.14. The 1-point Wightman function $W_1(x) = \langle\, \Omega, \phi(x)\Omega \,\rangle$ is a constant c by translation invariance, i.e. it is an element of R^*, and by invariance under rotations it is in $(R^*)^{\mathrm{Spin}_+(V)}$. Thus we may (and will) assume without loss of generality that $c = 0$ (otherwise we can replace $\phi(x)$ by $\phi(x) - c$). So we may assume without loss of generality that $W_1 = 0$.

Remark 11.15. The positivity property for the 2-point function can be written as

$$\int_{V^2} \mathbb{W}(x_2 - x_1)\overline{f(x_1)}f(x_2)dx_1dx_2 \geq 0,$$

where $\mathbb{W}(x) = W_2(0, x)$. Thus, taking Fourier transforms, we have

$$\int_V \widehat{\mathbb{W}}(p)\overline{\widehat{f}(p)}\widehat{f}(p)dp \geq 0.$$

This shows that $\widehat{\mathbb{W}}(p)dp$ is a measure concentrated on $\overline{V}_+$ and valued in nonnegative hermitian forms on $R_{\mathbb{C}}$.

11.9. **The mass spectrum of a Wightman QFT.** Let $\mathcal{H}^{(1)} \subset \mathcal{H}$ be the closure of the span of vectors $\phi(x)\Omega$, $x \in V$. It is called the space of 1-*particle states*, and it is clearly a $\widetilde{\mathbf{P}}$-subrepresentation of $\mathcal{H}$. The *mass spectrum* of the theory is determined by the structure of this representation. So we need to discuss the representation theory of $\widetilde{\mathbf{P}}$.

Since $\widetilde{\mathbf{P}}$ is a semidirect product, its irreducible unitary representations are unitarily induced. Namely, let $\mathcal{O}$ be an orbit of $\mathrm{Spin}_+(V)$ on V and ρ be an irreducible unitary representation of the stabilizer $\widetilde{\mathbf{P}}_0$ of a point $v_0 \in \mathcal{O}$. Then ρ defines an equivariant Hilbert bundle on $\mathcal{O}$ with total space $(\widetilde{\mathbf{P}} \times \rho)/\widetilde{\mathbf{P}}_0$ where $\widetilde{\mathbf{P}}_0$ acts diagonally. Thus we can consider the space $\mathcal{H}_{\mathcal{O},\rho}$ of square integrable half-densities on $\mathcal{O}$ with values in this bundle. This space carries a unitary representation of $\widetilde{\mathbf{P}}$. A theorem of Mackey then says that this unitary representation is irreducible, and all irreducible unitary representations of $\widetilde{\mathbf{P}}$ are obtained uniquely in this way. For example, if $\mathcal{O} = \{0\}$, then $\mathcal{H}_{0,\rho}$ is just a unitary irreducible representation of $\mathrm{Spin}_+(V)$.

Now we are ready to discuss the structure of the representation $\mathcal{H}^{(1)}$. By taking Fourier transforms (see Remark 11.15), we see that if $\mathcal{H}_{\mathcal{O},\rho}$ occurs in $\mathcal{H}^{(1)}$ then ρ needs to be finite dimensional. For example, for $d \geq 3$ and $\mathcal{O} = \langle 0 \rangle$ the only choice is the trivial representation, as the group $\mathrm{Spin}_+(V)$ is a connected semisimple non-compact Lie group. Moreover, if the theory has a unique vacuum then the trivial representation occurs in $\mathcal{H}$ discretely with multiplicity 1, as the span of the vacuum vector Ω. As $\mathcal{H}^{(1)}$ is orthogonal to Ω (since $W_1 = 0$), we see that the trivial representation does not occur in $\mathcal{H}^{(1)}$.

Let us now consider what happens with other orbits. By the positive energy condition, the only orbits that can occur are X_m^+ defined by $E = \sqrt{p^2 + m^2}$, $E > 0$ (where for $d = 2$ the set X_0^+ falls into two orbits X_0^{++} and X_0^{+-} defined by $p = \pm E > 0$). For $m > 0$ this is the upper sheet of a 2-sheeted hyperboloid and for $m = 0$ it is the upper part of the light cone (which is a union of two orbits for $d = 2$).

In the case $m > 0$, we may take $v_0 = (m, 0)$, then $\widetilde{\mathbf{P}}_0 = \mathrm{Spin}(d-1)$, so ρ is a (necessarily finite dimensional) unitary representation of this compact Lie group. Physicists say that this representation corresponds to a *massive particle of mass m and type ρ*. Particles arising in physically relevant quantum field theories are usually *scalars* ($\rho = \mathbb{C}$), *spinors* (ρ is a spinor representation of $\mathrm{Spin}(d-1)$) and *vectors* ($\rho = \mathbb{C}^{d-1}$ is the vector representation $\mathrm{Spin}(d-1)$). Note that by the spin-statistics theorem, scalars and vectors are bosons and spinors are fermions.

If $m = 0$, $d \geq 3$, then we can take $v_0 = (1, 1, 0, ..., 0)$, and the stabilizer is the non-reductive Lie group $\mathrm{Spin}(d-2) \ltimes \mathbb{R}^{d-2}$. Since ρ is finite dimensional, $\mathbb{R}^{d-2}$ has to act trivially, so ρ is an irreducible representation of the compact Lie group $\mathrm{Spin}(d-2)$. Physicists say that this representation corresponds to a *massless particle of type ρ*. The classification of massless particles is the same as for massive ones; however, note that since for massless particles ρ is a representation of $\mathrm{Spin}(d-2)$ rather than $\mathrm{Spin}(d-1)$, they in general have fewer components than massive ones; for example, a massless vector has one fewer component than a massive one.

If $m = 0, d = 2$ then there are two choices for v_0: $(1, 1)$ and $(1, -1)$. They have trivial stabilizer, so $\rho = \mathbb{C}$. Thus we have two types of massless particles: *right-moving* and *left-moving*, corresponding to the two choices of v_0. These particles are called this way since the corresponding operators $\phi(x)$ satisfy the conditions $\phi(t,x) = \phi(0, x-t)$, $\phi(t,x) = \phi(0,x+t)$, respectively, which classically would be right-moving and left-moving waves.

The set M of numbers m corresponding to representations $\mathcal{H}_{X_m^+,\rho}$ (or $\mathcal{H}_{X_0^{+\pm},\rho}$ for $d=2$) occurring in $\mathcal{H}^{(1)}$ is called the *mass spectrum* of the theory. One says that the theory has a *mass gap* when $\inf M = m > 0$. In this case the spectrum of $\widehat{H}$ is $\{0\} \cup [m, +\infty]$, so there is a gap between 0 and m. To find the mass spectrum, it suffices to look at the function $\widehat{\mathbb{W}}$: the mass spectrum is just the intersection of its support with the time axis (this follows from Remark 11.15).

11.10. **Free theory of a scalar boson.** Let us now construct a Wightman QFT corresponding to a scalar boson of mass $m > 0$. Recall that in the Lagrangian setting we had a 2-point function $G_M(x_2 - x_1)$, where $G_M(x)$ is a distribution satisfying the Klein-Gordon equation

$$(\Box + m^2) G_M = i\delta.$$

So at first sight for the corresponding Wightman QFT we want to have $\mathbb{W}(x) = G_M(x)$, so that the Lagrangian and Hamiltonian approach agree. However, the function $G_M(x)$ is even, while for $\mathbb{W}(x)$ we are supposed to have $\mathbb{W}(-x) = \overline{\mathbb{W}(x)}$, so our equality needs to be relaxed. In fact, the correct condition is that the identity $\mathbb{W}(x) = G_M(x)$ only needs to hold when x is spacelike or when $x \in \overline{V}_+$. When $x \in \overline{V}_-$, we should rather have $\mathbb{W}(x) = \overline{G_M(x)}$. In other words,

$$G_M(x_2 - x_1) = W_2^T(x_1, x_2)$$

is the so-called *time ordered* 2-point function, i.e. one obtained from $W_2(x_1, x_2)$ when x_1, x_2 are put in the chronological order (where in the spacelike separated case the order does not matter due to space locality).

We claim that with this definition the function $\mathbb{W}(x)$ satisfies the Klein-Gordon equation

$$(\Box + m^2)\mathbb{W} = 0$$

on the nose (without the delta-function on the right hand side). Indeed, we have $\mathrm{Re}\mathbb{W}(x) = \mathrm{Re}G(x)$, which satisfies the Klein-Gordon equation, so it remains to show that $\mathrm{Im}\mathbb{W}(x)$ satisfies it as well. But it is easy to see that $(\Box + m^2)\mathrm{Im}\mathbb{W}(x)$ is a distribution supported at the origin of homogeneity degree $-d$, so it is a multiple of δ. Since $\mathrm{Im}\mathbb{W}(x)$ is an odd function, this distribution must be zero, as claimed.

Also, since $G_M(x)$ is real for spacelike x, we get $\mathbb{W}(-x) = \overline{\mathbb{W}(x)}$. Thus the Fourier transform $\widehat{\mathbb{W}}(p)$ is real valued, supported on the hyperboloid X_m and invariant under $SO_+(V)$. It follows that

$$\widehat{\mathbb{W}}(p) = c_+\delta_{X_m^+} + c_-\delta_{X_m^-}.$$

where $c_\pm \in \mathbb{R}$. but in fact it can be shown that only $\delta_{X_m^+}$ occurs (this follows from the exponential decay of the Euclidean 2-point correlation function at infinity). Thus

$$\widehat{\mathbb{W}}(p) = c\delta_{X_m^+}.$$

In fact, one can show that $c = 2\pi$.

Similarly, we define higher W_n for $n > 2$ by the Wick formula, and this analysis implies after some work that these functions define a Wightman QFT.

In this case, $\mathcal{H}^{(1)} = L^2(X_m^+)$, so we have a single particle of mass m.

The theory of a free massless scalar, as well as massive and massless spinor is defined similarly.

11.11. **Normal ordering, composite operators and operator product expansion in a free QFT.** In classical field theory, given a classical scalar field $\phi(x)$, we may consider arbitrary polynomials and even any smooth functions of ϕ. The same is true for quantum mechanics, where $\phi(t)$ is a self-adjoint (possibly unbounded) operator on the Hilbert space $\mathcal{H}$ of quantum states, so using its spectral decomposition, we may define functions of ϕ. However, in quantum field theory in $d+1$ dimensions with $d \geq 1$ the situation is more complicated. Indeed, in this case $\phi(x)$ is not a usual operator-valued function of x, but rather a generalized one – an operator-valued distribution, and we know that for singular distributions, such as $\delta(x)$, we cannot even define the square $\delta(x)^2$.

Indeed, let $\phi(x)$ be a quantum scalar boson. Then the 2-point correlation function

$$\langle\phi(x)\phi(y)\rangle = \langle\Omega, \phi(x)\phi(y)\Omega\rangle = G(x-y)$$

blows up when $|x-y|^2 = 0$ (so in Euclidean signature, when $x = y$), so the operator $\phi^2(x)$ cannot possibly be well defined.

Thus, if we want to quantize the classical field $\phi^2(x)$, we need to regularize the corresponding operator product. This can be done by a standard regularization procedure called the *normally ordered product*.

For example, in Euclidean signature, the operator product $\phi(x)\phi(y)$ is well defined when $x \neq y$: indeed, by Wick's formula

$$\langle\phi(x)\phi(y)\phi(z_1)...\phi(z_k)\rangle =$$

$$G(x-y)\langle\phi(z_1)...\phi(z_k)\rangle+\sum_{i\neq j} G(x-z_i)G(y-z_j)\langle\phi(z_1)...\widehat{\phi}(z_i)...\widehat{\phi}(z_j)...\phi(z_k)\rangle,$$

where the hat indicates omissions (here $x, y, z_1, ..., z_k$ are distinct). Now, when $x \to y$, the first summand in this formula blows up while the second one does not. So it is natural to define the *normally ordered product* $:\phi(x)\phi(y):$ just by throwing away the singular terms, i.e. by the condition that its correlation function with $\phi(z_1)...\phi(z_k)$ is

$$\langle :\phi(x)\phi(y): \phi(z_1)...\phi(z_k)\rangle = \sum_{i\neq j} G(x-z_i)G(y-z_j)\langle\phi(z_1)...\widehat{\phi}(z_i)...\widehat{\phi}(z_j)...\phi(z_k)\rangle.$$

This is equivalent to just saying that

$$:\phi(x)\phi(y): \, = \phi(x)\phi(y) - G(x-y).$$

Note that while $\phi(x)\phi(y)$ blows up when $x = y$, the normally ordered product $:\phi(x)\phi(y):$ does not:

$$\langle :\phi^2(x): \phi(z_1)...\phi(z_k)\rangle = \sum_{i\neq j} G(x-z_i)G(x-z_j)\langle\phi(z_1)...\widehat{\phi}(z_i)...\widehat{\phi}(z_j)...\phi(z_k)\rangle.$$

This defines a *composite operator* $:\phi^2(x):$, which is a well defined operator-valued distribution.

Similarly one may define the normally ordered product $:\phi(x_1)...\phi(x_m):$ of any number of factors, by removing all the singular terms from the correlators. For example,

$$:\phi(x)\phi(y)\phi(z): \, = \phi(x)\phi(y)\phi(z) - G(x-y)\phi(z) - G(y-z)\phi(x) - G(z-x)\phi(y).$$

Such a product is well defined for all values of $x_1, ..., x_k$ and is commutative (independent of ordering of factors) and associative. We can also differentiate by x_j any number of times, to define the normally ordered product of arbitrary derivatives of ϕ. Evaluating such products on the diagonal (when all points are the same), we obtain composite operators attached to any differential monomials (hence polynomials) with respect to ϕ, such as $:\phi^3(x):$, $:\phi_{x_i}\phi_{x_j}:$, etc.

Exercise 11.16. *Derive a formula for the correlation function of several composite operators (evaluated at different points) in the theory of the scalar boson.*

In particular, we can now consider the product of two composite operators, e.g. $:\phi^2(x): \phi(y)$. Of course, this has a singularity at $x = y$, and an important problem is to understand the nature of this singularity. This is achieved by the procedure called the *operator product expansion*, which replaces the non-existent multiplication of composite operators.

To explain this procedure, consider first the simplest example of operator product:

$$\phi(x)\phi(y) = G(x-y) + :\phi(x)\phi(y): .$$

Using Taylor's formula, this can be rewritten so that the right hand side only contains $\phi(y)$ and no $\phi(x)$:

$$\phi(x)\phi(y) = G(x-y) + \sum_{\mathbf{n}} \frac{(x-y)^{\mathbf{n}}}{\mathbf{n}!} :\partial^{\mathbf{n}}\phi(y)\cdot\phi(y): ,$$

where $\mathbf{n} := (n_1, ..., n_{d+1})$, $(x-y)^{\mathbf{n}} := \prod_i (x_i - y_i)^{n_i}$, $\partial^{\mathbf{n}} := \prod_i \partial_{x_i}^{n_i}$, and $\mathbf{n}! := \prod_i n_i!$. In this sum, all terms except the first one are regular (i.e., continuous) at $x = y$.

Let us now try to write down a similar expansion for a more complicated example of operator product, $:\phi^2(x):\phi(y)$. We have

$$\langle :\phi^2(x):\phi(y)\phi(z_1)...\phi(z_k)\rangle = 2G(x-y)\langle\phi(x)\phi(z_1)...\widehat{\phi(z_j)}...\phi(z_k)\rangle +$$

$$\sum_{j,m,n \text{ distinct}} G(x-z_j)G(x-z_m)G(y-z_n)\langle\phi(z_1)...\widehat{\phi(z_j)}...\widehat{\phi(z_m)}...\widehat{\phi(z_n)}...\phi(z_k)\rangle.$$

Thus we get

$$:\phi^2(x):\phi(y) = 2G(x-y)\phi(y) + :\phi^2(x)\phi(y): .$$

As before, using Taylor's formula, this can be rewritten so that the right hand side only contains $\phi(y)$ and no $\phi(x)$:

$$:\phi^2(x):\phi(y) = 2G(x-y)\phi(y) + \sum_{\mathbf{n},\mathbf{m}} \frac{(x-y)^{\mathbf{n}+\mathbf{m}}}{\mathbf{n}!\mathbf{m}!} :\partial^{\mathbf{n}}\phi(y)\cdot\partial^{\mathbf{m}}\phi(y)\cdot\phi(y): .$$

And again, all terms except the first one are regular at $x = y$.

As a final example, consider the product $:\phi^2(x):\cdot:\phi^2(y):$. A similar computation yields

$$:\phi^2(x):\cdot:\phi^2(y): = 2G^2(x-y) + 4G(x-y) :\phi(x)\phi(y): + :\phi^2(x)\phi^2(y): ,$$

and as before we can expand this to remove $\phi(x)$ using Taylor's formula. Namely, expanding the second summand, we get

$$:\phi^2(x):\cdot:\phi^2(y): =$$

$$2G^2(x-y) + 4G(x-y)\sum_{\mathbf{n}} \frac{(x-y)^{\mathbf{n}}}{\mathbf{n}!} :\partial^{\mathbf{n}}\phi(y)\cdot\phi(y): + :\phi^2(x)\phi^2(y): ,$$

and the last summand can be expanded similarly. We now see that there are many singular terms: $G(x)$ behaves as $|x|^{1-d}$ for $d > 1$ and as $\log|x|$ for $d = 1$, so the singular terms are the ones with $|\mathbf{n}| \leq d-1$, where $|\mathbf{n}| := \sum_i n_i$. For example, for $d = 2$ for the massless boson we have

$$: \phi^2(x) : \cdot : \phi^2(y) : =$$

$$\frac{2}{|x-y|^2} + \frac{4}{|x-y|} : \phi^2(y) : + \sum_{j=1}^{3} \frac{4}{|x-y|}(x_j - y_j) : \partial_{x_i}\phi(y) \cdot \phi(y) : + \text{ regular.}$$

Yet we see that the number of singular terms is finite. In fact, it is not hard to prove the following proposition (see [QFS], vol 1, p.449).

Proposition 11.17. *Let A, B be two composite operators in the theory of scalar boson. Then there exist a unique collection of functions $F_j(y)$ and composite operators $C_j(y)$ such that we have an asymptotic expansion*

$$A(x)B(y) \sim \sum_j F_j(x-y)C_j(y), \ x \to y$$

such that for every N we have $|F_j(z)| = O(|z|^N)$, $z \to 0$, for all but finitely many j. In particular, there are finitely many singular terms (not continuous at $x = y$).

The expansion of Proposition 11.17 is called the *operator product expansion*. It is not hard to show that it exists in any free quantum field theory.

11.12. **Symmetries in quantum field theory.** In studying any physical system, it is crucial to find all its symmetries and use them to their full potential. For example, the equations of motion of a particle in a rotationally symmetric potential field can be fully solved by utilizing the rotational symmetry (see [A]).

The most fundamental fact about symmetries in classical or quantum mechanics is that for any 1-parameter group of symmetries of the system there is an (essentially unique) observable responsible for this symmetry, which is conserved in this system; i.e., every 1-parameter symmetry corresponds to a conservation law, and vice versa. This statement is called *Noether's theorem*.

Let us first explain the precise meaning of Noether's theorem in the setting of classical mechanics. Suppose we have a system with phase space a symplectic manifold (M, ω) (typically $M = T^*X$, where X is the configuration space, and $\omega = d\alpha$ is the differential of the Liouville form) and hamiltonian $H \in C^\infty(M)$. Let g^t be a 1-parameter group of symmetries of this system, i.e., of symplectic diffeomorphisms of M which preserve H. Let $v := \frac{d}{dt}|_{t=0} g^t$ be the vector field generating the flow g^t. Then we have $L_v\omega = 0$ (i.e., v is a symplectic vector field, so $\omega_v := \omega(v, ?)$ is a closed 1-form), and $L_v H = 0$. Let us assume that M is simply connected (for example, we can restrict ourselves to a neighborhood of a point in M or X). In this case ω_v is exact, so there exists $Q \in C^\infty(M)$ (unique up to adding a constant) such that $\omega_v = dQ$. Then for any observable $F \in C^\infty(M)$ we have $L_v F = \{Q, F\}$. Moreover $\{Q, H\} = L_v H = 0$. The observable Q is thus conserved under the hamiltonian flow and is the conservation law corresponding to the 1-parameter group g^t. It is called (especially in the setting of field theory) the *Noether (conserved) charge* of the symmetry.

A trivial example of this is the hamiltonian flow h^t defined by the hamiltonian H itself, i.e., the time translation symmetry; in this case $Q = H$, so the corresponding conserved quantity is H (the energy). Other examples include the momenta $p_1, ..., p_n$ which corresponds to translation symmetry (for $X = \mathbb{R}^n$) and *angular momenta* $M_{kj} := x_k p_j - x_j p_k$ corresponding to rotational symmetries around the codimension 2 hyperplanes $x_k = x_j = 0$.

More generally, suppose G is a Lie group acting (on the right) by symmetries of the system. Let $\mathfrak{g} = \mathrm{Lie}G$ be the Lie algebra of G. Any element $y \in \mathfrak{g}$ gives rise to a 1-parameter subgroup $e^{ty} \in G$, so defines a conserved quantity Q_y such that

$$\{Q_y, F\} = y \cdot F := \tfrac{d}{dt}|_{t=0} e^{ty} \cdot F$$

for each $F \in C^\infty(M)$, where $(g \cdot F)(m) = F(mg)$, $m \in M$. More precisely, Q_y is defined only up to adding a constant, so let us fix some linear assignment $y \mapsto Q_y$.

Moreover, it is clear that for $y, z \in \mathfrak{g}$

$$\{Q_y, Q_z\} = Q_{[y,z]} + C(y, z),$$

where $C(y, z)$ is a skew-symmetric bilinear form on $\mathfrak{g}$ which arises because Q_y is uniquely determined by y only up to adding a constant. Furthermore, by the Jacobi identity, the form C is a 2-cocycle :

$$C([x, y], z) + C([y, z], x) + C([z, x], y) = 0.$$

It follows that the assignment $y \mapsto Q_y$ is almost a homomorphism $\mathfrak{g} \to C^\infty(M)$, but not quite: rather, it defines a homomorphism

$$\mu : \widehat{\mathfrak{g}} \to C^\infty(M),$$

where $\widehat{\mathfrak{g}} := \mathfrak{g} \oplus \mathbb{R}$ is a 1-dimensional central extension of $\mathfrak{g}$ with commutator

$$[(y, a), (z, b)] = ([y, z], C(y, z)).$$

Namely, $\mu(y, a) = Q_y + a$.

The map μ may be viewed as an element of $C^\infty(M)\otimes\widehat{\mathfrak{g}}^*$, i.e., geometrically as a C^∞-map

$$\mu: M\to\widehat{\mathfrak{g}}^*.$$

This map is called the *moment map* and plays a fundamental role in symplectic geometry.

The following example shows that the cohomology class of C may be nonzero, which means that we may not be able to choose Q_y to make $C=0$.

Example 11.18. The group $\mathbb{R}^{2n}$ acts on $M=T^*\mathbb{R}^n$ (with trivial hamiltonian $H=0$) by translations. So we have $\mathfrak{g}=\mathbb{R}^{2n}$ and $C(y,z)=\omega(y,z)$. Thus $\widehat{\mathfrak{g}}$ is the *Heisenberg Lie algebra* $\mathbb{R}^{2n}\oplus\mathbb{R}$ with commutation relations

$$[(y,a),(z,b)]=([y,z],C(y,z)),$$

which is a non-trivial central extension of $\mathfrak{g}$.

However, in many examples the cohomology class $[C]\in H^2(\mathfrak{g})$ is, in fact, zero, i.e., $\widehat{\mathfrak{g}}=\mathfrak{g}\oplus\mathbb{R}$ as Lie algebras. For instance, this is automatically so if $H^2(\mathfrak{g})=0$ (e.g., if G is a compact Lie group). In this case, we may choose Q_y so that $C=0$, and we have a moment map

$$\mu: M\to\mathfrak{g}^*.$$

For example, for translation symmetries of the free particle, μ is the momentum $\mathbf{p}$ of the particle, which explains the terminology "moment map".

A similar discussion applies to classical field theory, using the formalism of Subsection 11.6. Namely, in this case, the Noether charge is given by the integral over the space of a certain local field called *Noether current*.

For example, consider the free massive boson ϕ on the spacetime $\mathbb{R}^d\times\mathbb{R}$. The Hamiltonian is

$$H=\frac{1}{2}\int_X(\phi_t^2+|d_x\phi|^2+m^2\phi^2)dx.$$

Thus $H=\int_{\mathbb{R}^d}Jdx$ where

$$J=\frac{1}{2}(\phi_t^2+|d_x\phi|^2+m^2\phi^2)=\frac{1}{2}(:\phi_t^2:+\sum_{j=1}^{d}:\phi_{x_j}^2:+m^2:\phi^2:) \quad (11.3)$$

is the Noether current associated to the time translation symmetry.

Similarly, the Noether current for the spacial translation in the i-th coordinate is

$$J_k=\phi_t\phi_{x_k}. \quad (11.4)$$

Indeed, using the formulas of Subsection 11.6, we have

$$\{J_k(x), \phi(y)\} = -\phi_{x_k}(x)\delta(x-y),\ \{J_k(x), \phi_t(y)\} = \phi_t(x)\delta_{x_k}(x-y).$$

Thus defining the charge

$$P_k = \int_{\mathbb{R}^d} J_k(x)dx,$$

using integration by parts, we get

$$\{P_k, \phi(y)\} = -\phi_{x_k}(y),\ \{P_k, \phi_t(y)\} = -\phi_{tx_k}(y),$$

as needed.

Furthermore, this discussion extends to quantum theory, with observables replaced by operators as usual. Namely, in this case, we have a unitary projective representation $\pi : G \to \mathrm{Aut}(\mathcal{H})$ of the Lie group G of symmetries on the Hilbert space $\mathcal{H}$ of quantum states of the system, so that $[\pi(g), \widehat{H}] = 0$, where $\widehat{H} : \mathcal{H} \to \mathcal{H}$ is the hamiltonian (an unbounded self-adjoint operator). The quantum Noether charges corresponding to these symmetries simply define the corresponding Lie algebra representation $\pi_* : \mathfrak{g} \to \mathrm{End}(\mathcal{S})$, where $\mathcal{S}$ is a certain dense subspace of $\mathcal{H}$ (of smooth vectors) on which all the operators $\pi_*(y)$ are defined. For instance, in quantum mechanics, like in classical one, the time translation corresponds to the Hamiltonian $\widehat{H}$, the spacial translations to the momentum operators $\widehat{p}_j := -i\hbar\partial_{x_j}$, and rotations around $x_k = x_j = 0$ to the angular momentum operators

$$\widehat{M}_{kj} := -i\hbar(x_k\partial_j - x_j\partial_k).$$

Finally, in quantum field theory, by analogy with classical one, a quantum Noether charge is an operator of the form

$$Q = \int_{\mathbb{R}^d} J(x)dx,$$

where $J(x)$ is a quantum local operator called the *quantum Noether current.* For example, in the case of a free massive boson, the currents $J(x)$ and $J_k(x)$ for time and space translations are given by the same formulas (11.3),(11.4), but now with $\phi(x,t)$ being the quantum field corresponding to the massive boson (say, in the setting of Wightman axioms) rather than the classical field, and with normal ordered product instead of the usual product:

$$J = \frac{1}{2}(:\phi_t^2: + \sum_{j=1}^{d} :\phi_{x_j}^2: + m^2 :\phi^2:),$$

$$J_k =: \phi_t\phi_{x_k}: ,$$

and the corresponding charges, as in the classical case, are given by integration of the current over the space. For example, for the free boson

$$\widehat{H} = \int_{\mathbb{R}^d} J(x)dx \tag{11.5}$$

is the quantum hamiltonian, and

$$\widehat{P}_k := \int_{\mathbb{R}^d} J_k(x)dx$$

are the quantum momentum operators.

11.13. **Field theories on manifolds.** As already mentioned above, an important feature of classical and quantum field theory is the possibility to consider them not just on a Euclidean or Minkowskian space, but more generally on Riemannian and Lorentzian manifolds. The main examples are theories on $X \times \mathbb{R}$, where X is a Riemannian d-dimensional space manifold with metric $g_{ij}dx^i dx^j$ (Einstein summation) and $\mathbb{R}$ is the time line, with Lorentzian metric

$$|dx|^2 := (dt)^2 - g_{ij}dx^i dx^j,$$

and Euclidean theories on a Riemannian $d+1$-dimensional spacetime manifold M.

Here we will consider only classical field theories on manifolds. These theories can then be quantized using either Lagrangian or Hamiltonian approach, but we will not discuss this, except in some examples. The story is parallel to the case of flat space considered above, but we should make sure that the kinetic term and other terms in the Lagrangian are defined canonically (i.e., do not depend on the choice of coordinates). For simplicity consider the Euclidean case (in the Lorentzian case the story is similar). We restrict ourselves to reviewing the most common types of classical fields in such theories, as well as the corresponding kinetic and other terms in their Lagrangians. A more complete discussion can be found in [QFS].

1. **Scalar (bosonic) fields.** In the simplest case a scalar field is just a real function on M (real scalar), but one can also consider scalars valued in a finite dimensional real vector space with a positive inner product (for example, $\mathbb{C}$, for complex scalars) or, more generally, valued in a real vector bundle on M. The kinetic term for a scalar $\phi : M \to E$ valued in a vector space $E \cong E^*$ with inner product is $|d\phi(x)|^2$, the squared norm of the vector $d\phi \in T_{\phi(x)}M \otimes E$ with respect to the inner products on $T_{\phi(x)}M$ and E. Thus if this vector has components $(d\phi)_{ij}$

in orthonormal bases then

$$|d\phi|^2 = \sum_{i,j} (d\phi)_{ij}^2.$$

More generally, if E is a vector bundle on M then we need to fix an inner product on E (i.e., E should be an orthogonal bundle) and also a connection A preserving this inner product, which gives rise to the covariant derivative operator ∇_A; if E is trivialized on a local chart $U \subset M$ then A becomes a 1-form on U with values in $\mathfrak{o}(E)$ and we have $\nabla_A = d + A$. In this case, an E-valued scalar field ϕ is a section of E over M, and the kinetic term is $|\nabla_A \phi|^2$, which in local trivialization has the form $|d\phi + A\phi|^2$.

Note that for a scalar field ϕ, we can always add to the kinetic term a mass term $m^2|\phi|^2$, where m^2 is a real number. More generally, we can add a mass term $(\phi, Q\phi)$, where Q is a self-adjoint endomorphism of E.

2. **Spinor (fermionic) fields.** Spinor fields can be defined on a spin manifold M, i.e., an oriented manifold equipped with a spin structure (a lift of the tangent bundle from $SO(n)$ to $\mathrm{Spin}(n)$). For such a manifold, we have the canonically defined *spin bundle* S_M, which is the associated bundle to the above $\mathrm{Spin}(n)$ bundle via the spin representation $\mathrm{Spin}(n) \to \mathrm{Aut}(S)$. This bundle carries a natural inner product and a connection induced by the Levi-Civita connection of M that preserves this inner product. Moreover, as explained in Subsection 11.3, in even dimensions we have $S = S_+ \oplus S_-$, where S_+, S_- are irreducible representations of $\mathrm{Spin}(n)$, so we have $S_M = S_{M+} \oplus S_{M-}$, an orthogonal decomposition of S_M into two subbundles.

Spinor fields, in the most basic case, are sections of the Spin bundle S_M. The sections of S_{M+} and S_{M-}, as noted in Subsection 11.4, are called *chiral spinors*.

The possible kinetic and mass terms for spinors on the flat space are described in Subsection 11.4, and the story on the curved manifold is similar. The only new feature is that we have to define the Dirac operator $\mathbf{D}$ for a spinor field on an arbitrary spin manifold. To this end, all we have to do is replace ordinary partial derivatives in formula (11.2) by the covariant ones with respect to the Levi-Civita connection:

$$\mathbf{D} = \sum_i \Gamma_i \nabla_i^{LC}. \tag{11.6}$$

More generally, similar to the scalar field case, we may consider spinors valued in a vector bundle E with an inner product and an orthogonal connection A, i.e., sections of the bundle $S_M \otimes E$. This bundle carries a tensor product connection $\nabla^{\rm total} = \nabla^{LC} \otimes \nabla_A$, and the Dirac operator is defined by the formula

$$\mathbf{D} = \sum_i \Gamma_i \nabla_i^{\rm total}.$$

3. **Gauge fields.** Let G be a compact Lie group $\mathfrak{g} = {\rm Lie}G$ equipped with a positive invariant inner product. Gauge fields are connections A on principal G-bundles E on M, so in local trivialization A is a 1-form on M with values in $\mathfrak{g}$ and the covariant derivative with respect to A looks like $\nabla_A = d + A$. The connection A has curvature F_A (called *field strength* in physical terminology), which is a 2-form on M with values in the adjoint bundle ${\rm ad}E$. In local trivialization the curvature of A is the Maurer-Cartan form

$$F_A = dA + \frac{1}{2}[A, A].$$

In particular, if G is abelian then we just have $F_A = dA$. The kinetic term for a gauge field A is $|F_A|^2$, where the squared norm is taken with respect to the inner product on $(\wedge^2 T^*M \otimes {\rm ad}E)_x$ induced by the inner products on T_xM and $\mathfrak{g}$ (note that this does not depend on the identification of Lie algebras $({\rm ad}E)_x \cong \mathfrak{g}$ since the form on $\mathfrak{g}$ is invariant).

It makes sense to fix the topological type of the C^∞-bundle E (which does not change under deformations) and consider the space ${\rm Conn}(E)$ of all connections A on E. If $A_1, A_2 \in {\rm Conn}E$ then $\nabla_{A_1} - \nabla_{A_2} \in \Omega^1(M) \otimes {\rm ad}E$, so ${\rm Conn}(E)$ is an affine space with underlying vector space $\Omega^1(M) \otimes {\rm ad}E$. Moreover, this space carries a natural right affine linear action of the gauge group $\mathcal{G}_E = C^\infty(M, E)$, which in local trivialization looks like

$$A^g = g^{-1}dg + g^{-1}Ag.$$

The configuration space of a classical gauge theory is then

$$\mathcal{M} := \sqcup_{\text{topological types } E} {\rm Conn}(E)/\mathcal{G}_E,$$

so the phase space is the cotangent bundle $T^*\mathcal{M}$.