Fall 2013

18.310 Homework 4 Solutions

Instructions: Remember to submit a separate PDF for each question. Do not forget to include a list of your collaborators or to state that you worked on your own.

1. Let $(a_n)_{n\geq 0}$ be the sequence defined by $a_0 = 0$, $a_1 = 5$ and $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$. Find an explicit expression for a_n.

   Solution: We calculate the generating function

   $$A(x) = \sum_{n\geq 0} a_n x^n$$

   of a_n. Since $a_0 = 0$ and $a_1 = 5$,

   $$A(x) = 5x + \sum_{n\geq 2} a_n x^n.$$

   Using that $a_n = a_{n-1} + 6a_{n-2}$, we get

   $$A(x) = 5x + xA(x) + 6x^2 A(x)$$

   so that

   $$A(x) = \frac{5x}{1 - x - 6x^2} = \frac{1}{1-3x} - \frac{1}{1+2x}.$$

   This can be rewritten as a power series

   $$A(x) = \sum_{n\geq 0} (3^n - (-2)^n) x^n.$$

   Identifying like terms in the definition and this last expression we get

   $$a_n = 3^n - (-2)^n,$$

   for all n. To make sure we haven't made any mistakes, we check a few values and indeed $a_0 = 0$, $a_1 = 5$ and $a_2 = 5$.

2. Given some $r \in \mathbb{R}$, consider the generating function $F(x) = \frac{1}{(1-rx)^2}$ corresponding to a sequence $(f_n)_{\geq 0}$. Find an explicit expression for f_n.

   Solution: Here is a derivation:

   $$F(x) = \left(\frac{1}{1-rx}\right)^2 = \left(\sum_{i\geq 0} (rx)^i\right)^2 = \sum_{n\geq 0} \sum_{i+j=n} (rx)^i (rx)^j = \sum_{n\geq 0} (n+1) r^n x^n.$$

   So $f_n = (n+1)r^n$.

3. A binary number consists of a sequence of 0's and 1's, such as 0100, 110001 or even the empty sequence $\cdot$ containing no bits. Let F denote the set of all pairs (a, b) where a and b are binary numbers. Let d_n be the number of pairs in F having a total of n bits. Thus $d_0 = 1$ as it corresponds only to $(\cdot, \cdot)$, while $d_1 = 4$ as there are 4 pairs with a total of 1 bit: $(0, \cdot), (1, \cdot), (\cdot, 0), (\cdot, 1)$.

   (a) Find the generating function for $(d_n)_{n\geq 0}$

   (b) Derive from it an explicit formula for d_n.

   (c) Explain how you could have derived this expression directly without considering its generating function.

   Solution:

   (a) The class of objects described is the class of binary strings times itself (Cartesian product), so the generating function F for d_n is just the square of the generating function for the number of n bit binary strings, i.e.

$$F(x) = \left(\sum_{n\geq 0} (2x)^n \right)^2 = \left(\frac{1}{1-2x} \right)^2.$$

   (b) By question 2, we deduce that $d_n = (n+1)2^n$.

   (c) Note that the number d_n is just the number of ways to choose an n bit string, and put a comma in one of the $n+1$ spaces between bits (including the space before everything, and after everything). There are exactly $(n+1)2^n$ ways of doing this.