2. K-finite vectors and matrix coefficients

2.1. **K-finite vectors.** Let K be a *compact* topological group. In this case K has a unique right-invariant Haar measure of volume 1, which is therefore also left-invariant; we will denote this measure by dg. Thus if V is a finite dimensional (continuous) representation of K and B a positive definite Hermitian form on V then the form

$$\overline{B}(v,w) := \int_K B(gv, gw)dg$$

is positive definite and K-invariant, which implies that V is unitary. If V is irreducible then by Schur's lemma this unitary structure is unique up to scaling.

This implies that finite dimensional representations of K are completely reducible: if $W \subset V$ is a subrepresentation then $V = W \oplus W^{\perp}$, where $W^{\perp}$ is the orthogonal complement of W under the Hermitian form.

Now let V be any continuous representation of K (not necessarily finite dimensional).

Definition 2.1. A vector $v \in V$ is **K-finite** if it is contained in a finite-dimensional subrepresentation of V. The space of K-finite vectors of V is denoted by V^{fin}.

Let $\mathrm{Irr}K$ be the set of isomorphism classes of irreducible finite dimensional representations of K. We have a natural K-invariant linear map

$$\xi : \oplus_{\rho\in \mathrm{Irr}K} \mathrm{Hom}(\rho, V) \otimes \rho \to V^{\mathrm{fin}}$$

(where K acts trivially on $\mathrm{Hom}(\rho, V)$) defined by

$$\xi(h \otimes u) = h(u).$$

Lemma 2.2. *ξ is an isomorphism.*

Proof. To show ξ is injective, assume the contrary, and let $\widetilde{\rho}$ be an irreducible subrepresentation of $\mathrm{Ker}\xi$. Then $\widetilde{\rho} = h \otimes \rho$, so for any $u \in \rho$, $h(u) = \xi(h \otimes u) = 0$, so $h = 0$, contradiction.

It remains to show that ξ is surjective. For $v \in V^{\mathrm{fin}}$, let $W \subset V^{\mathrm{fin}}$ be a finite dimensional subrepresentation of V containing v. By complete reducibility, W is a direct sum of irreducible representations. Thus it suffices to assume that W is irreducible. Let $h : W \hookrightarrow V$ be the corresponding inclusion. Then $v = h(v) = \xi(h \otimes v)$. □

Example 2.3. Let $K = S^1 = \mathbb{R}/2\pi\mathbb{Z}$. The irreducible finite dimensional representations of K are the characters $\rho_n(x) = e^{inx}$ for integer n. Let $V = L^2(S^1)$. Then $\mathrm{Hom}(\rho_n, V)$ is the space of functions on S^1

such that $f(x+a) = e^{ina}f(x)$, which is a 1-dimensional space spanned by the function e^{inx}. It follows that V^{fin} is the space of trigonometric polynomials $\sum_n a_n e^{inx}$, where only finitely many coefficients $a_n \in \mathbb{C}$ are nonzero.

2.2. **Matrix coefficients.** Let us now consider the special case $V = L^2(K)$, and view it as a representation of $K \times K$ via

$$(\pi(a,b)f)(x) = f(a^{-1}xb).$$

For every irreducible representation $\rho \in \mathrm{Irr}K$ we have a homomorphism of representations of $K \times K$:

$$\xi_\rho : \mathrm{End}_{\mathbb{C}}\rho = \rho^* \otimes \rho \to L^2(K)$$

defined by

$$\xi_\rho(h \otimes v)(g) := h(gv).$$

This map is nonzero, hence injective (as $\rho^* \otimes \rho$ is an irreducible $K \times K$-module), and is called **the matrix coefficient map**, as the right hand side is a matrix coefficient of the representation ρ. The **theorem on orthogonality of matrix coefficients** tells us that the images of ξ_ρ for different ρ are orthogonal, and for $A, B \in \mathrm{End}_{\mathbb{C}}\rho$ we have

$$(\xi_\rho(A), \xi_\rho(B)) = \frac{\mathrm{Tr}(AB^\dagger)}{\dim \rho},$$

where $B^\dagger$ is the Hermitian adjoint of B with respect to the unitary structure on ρ. Thus, choosing orthonormal bases $\{v_{\rho i}$ in each ρ, we find that the functions

$$\psi_{\rho ij} := (\dim \rho)^{\frac{1}{2}} \xi_\rho(E_{ij}),$$

where $E_{ij} := v_{\rho j}^* \otimes v_{\rho i}$ are elementary matrices, form an orthonormal system in $L^2(K)$.

Let us view $L^2(K)$ as a representation of K via left translations. Let $\rho \in \mathrm{Irr}K$. Then every $h \in \rho$ defines a homomorphism of representations $f_h : \rho^* \to L^2(K)$ which, when viewed as an element of $L^2(K, \rho)$, is given by the formula $f_h(y) := yh$. Conversely, suppose $f : \rho \to V$ is a homomorphism. Then f can be represented by an L^2-function $\widetilde{f} : K \to \rho$ such that for any $b \in K$, the function $x \mapsto \widetilde{f}(bx) - b\widetilde{f}(x)$ vanishes outside a set $S_b \subset K$ of measure 0. Let $S \subset K \times K$ be the set of pairs (b, x) such that $x \in S_b$. Then S has measure 0, hence the set T_x of $b \in K$ such that $(b, x) \in S$ (i.e., $x \in S_b$) has measure zero almost everywhere with respect to x. So pick $x \in K$ such that T_x has measure zero. For $y = bx \notin T_x x$, we have $x \notin S_b$, so $\widetilde{f}(y) = yx^{-1}\widetilde{f}(x)$. Thus

$f = f_h$ where $h = x^{-1}\widetilde{f}(x)$. It follows that the assignment $h \mapsto f_h$ is an isomorphism $\rho \cong \mathrm{Hom}(\rho^*, L^2(K))$. This shows that the map

$$\bigoplus_{\rho\in \mathrm{Irr}K} \xi_\rho : \bigoplus_{\rho\in \mathrm{Irr}K} \rho^* \otimes \rho \to L^2(K)^{\mathrm{fin}}$$

is an isomorphism, where $L^2(K)^{\mathrm{fin}}$ is the space of K-finite vectors in $L^2(K)$ under left translations. Thus any K-finite function under left (or right) translations is actually $K \times K$-finite, and we have a natural orthogonal decomposition

$$L^2(K)^{\mathrm{fin}} \cong \bigoplus_{\rho\in \mathrm{Irr}K} \rho^* \otimes \rho.$$

Moreover, since $L^2(K)$ is separable, it follows that $\mathrm{Irr}K$ is a countable set.

2.3. **The Peter-Weyl theorem.** The following non-trivial theorem is proved in the basic Lie groups course.

Theorem 2.4. *(Peter-Weyl) $L^2(K)^{\mathrm{fin}}$ is a dense subspace of $L^2(K)$. Hence $\{\psi_{\rho ij}\}$ form an orthonormal basis of $L^2(K)$, and we have*

$$L^2(K) = \widehat{\oplus}_{\rho\in \mathrm{Irr}K} \rho^* \otimes \rho.$$

(completed orthogonal direct sum under the Hilbert space norm).

Example 2.5. For $K = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ the Peter-Weyl theorem says that the Fourier system $\{e^{inx}\}$ is complete, i.e., a basis of $L^2(S^1)$.

2.4. **Partitions of unity.** Let X be a metric space with distance function d, and $C \subset X$ a closed subset. For $x \in X$ define

$$d(x, C) := \inf_{y\in C} d(x, y)$$

if $C \neq \emptyset$. This function is continuous, since $d(x, C) \le d(x, y) + d(y, C)$, hence $|d(x, C) - d(y, C)| \le d(x, y)$. Thus the function $f_C(x) := \frac{d(x,C)}{1+d(x,C)}$ (defined to be 1 if $C = \emptyset$) is continuous on X, takes values in $[0, 1]$, and $f_C(x) = 0$ iff $x \in C$. So if $\{U_i, i \in \mathbb{N}\}$ is a countable open cover of X then the function $\sum_{i\in\mathbb{N}} 2^{-i} f_{U_i^c}$ is continuous and strictly positive, so we may define the continuous functions on X

$$\phi_i := \frac{2^{-i} f_{U_i^c}}{\sum_{i\in\mathbb{N}} 2^{-i} f_{U_i^c}}, i \in \mathbb{N}$$

These functions form a **partition of unity subordinate to the cover** $\{U_i, i \in \mathbb{N}\}$: each ϕ_i is non-negative, vanishes outside U_i, and $\sum_{i\in\mathbb{N}} \phi_i = 1$ (a uniformly convergent series on X).

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