

4. Plancherel formulas, Dirac sequences, smooth vectors

4.1. Plancherel formulas. For a compactly supported L^1 -function f on G , for brevity let us denote $\pi(fdg)$ just by $\pi(f)$.

Proposition 4.1. (*Plancherel's theorem for compact groups*) *Let K be a compact group and $f_1, f_2 \in L^2(K)$. Then*

$$(f_1, f_2) = \sum_{\rho \in \text{Irr}K} \dim \rho \cdot \text{Tr}(\pi_\rho(f_1)\pi_\rho(f_2)^\dagger)$$

and this series is absolutely convergent.

Proof. Recall that if e_i is an orthonormal basis of a Hilbert space H and $f_1, f_2 \in H$ then

$$(f_1, f_2) = \sum_i (f_1, e_i)(e_i, f_2)$$

and this series is absolutely convergent. The result now follows by applying this formula to the orthonormal basis provided by the Peter-Weyl theorem:

$$\psi_{\rho ij} = \sqrt{\dim \rho}(\pi_\rho(g)v_{\rho i}, v_{\rho j}),$$

where $\{v_{\rho i}\}$ is an orthonormal basis of ρ . □

Example 4.2. If $K = S^1$, Plancherel's theorem reduces to the usual Parseval equality in Fourier analysis:

$$(f_1, f_2) = \sum_{n \in \mathbb{Z}} c_n(f_1)\overline{c_n(f_2)},$$

where $c_n(f)$ are the Fourier coefficients of f .

Proposition 4.3. (*Plancherel's formula*) *If K is a compact Lie group and $f \in C^\infty(K)$ then*

$$f(1) = \sum_{\rho \in \text{Irr}K} \dim \rho \cdot \text{Tr}(\pi_\rho(f))$$

and this series is absolutely convergent.

Example 4.4. If $K = S^1$ then this formula says that for $f \in C^\infty(S^1)$

$$f(1) = \sum_{n \in \mathbb{Z}} c_n(f),$$

i.e., the Fourier series of f absolutely converges at 1. Note that for $f \in C(S^1)$ this is false in general!¹⁰

¹⁰One can show that for an N -dimensional group, the differentiability needed for the Plancherel formula is C^k for $k > N/2$.

Proof. Consider the integral operator A of convolution with the function f :

$$(A\psi)(x) = (f * \psi)(x) = \int_K f(xy^{-1})\psi(y)dy.$$

This operator is trace class, since it has smooth integral kernel $F(x, y) = f(xy^{-1})$, and

$$\text{Tr}(A) = \int_K F(x, x)dx = \int_K f(1)dx = f(1).$$

On the other hand, A is right-invariant, so it preserves the decomposition of $L^2(K)$ into the direct sum of $\rho \otimes \rho^*$ and acts on each such summand as $\pi_\rho(f) \otimes 1$. Thus we also have

$$\text{Tr}(A) = \sum_{\rho \in \text{Irr}K} \dim(\rho) \cdot \text{Tr}(\pi_\rho(f)),$$

as desired. □

4.2. Dirac sequences. If G is a locally compact group then multiplication by dg defines an inclusion $C_c(G) \hookrightarrow \text{Meas}_c(G)$ of compactly supported continuous functions into compactly supported measures as a (non-unital) subalgebra. Moreover, if G is a Lie group then we have a nested sequence of subalgebras $C_c^k(G)$, $0 \leq k \leq \infty$ (compactly supported C^k -functions). The following lemma shows that while these subalgebras are non-unital, they are “almost unital”.

Lemma 4.5. *There exists a sequence $\phi_n \in C_c(G)$ such that $\phi_n \rightarrow \delta_1$ in the weak topology as $n \rightarrow \infty$. Moreover, if G is a Lie group, we can choose $\phi_n \in C_c^\infty(G)$.*

Proof. (sketch) ϕ_n can be constructed as a sequence of “hat” functions supported on a decreasing sequence of balls $B_1 \supset B_2 \supset \dots$ whose intersection is $1 \in G$. Such hat functions can be chosen smooth if G is a Lie group. □

Such sequences ϕ_n are called **Dirac sequences**.

Corollary 4.6. *$C_c(G)$ is sequentially dense in $\text{Meas}_c(G)$. For Lie groups, $C_c^\infty(G)$ is sequentially dense in $\text{Meas}_c(G)$.*

Proof. By translating a Dirac sequence, for any $g \in G$ we can construct a sequence $\psi_n \rightarrow \delta_g$. This implies that $\text{Meas}_c^0(G)$ is contained in the sequential closure of $C_c(G)$ (and of $C_c^\infty(G)$ in the Lie case). So the result follows from Lemma 3.4. □

4.3. Density of K -finite vectors.

Corollary 4.7. *Let V be a continuous representation of a compact group K . Then V^{fin} is dense in V .*

Proof. Let $v \in V$, and $\phi_n \rightarrow \delta_1$ a continuous Dirac sequence, which exists by Lemma 4.5. Then $\pi(\phi_n)v \rightarrow v$ as $n \rightarrow \infty$. But $\phi_n \in L^2(K)$, so by the Peter-Weyl theorem, there exists $\psi_n \in L^2(K)^{\text{fin}} = \oplus_{\rho} \rho^* \otimes \rho$ such that

$$\|\psi_n - \phi_n\|_2 < \frac{1}{n}.$$

Then $\psi_n - \phi_n \rightarrow 0$ in $L^2(K)$, hence in $\text{Meas}_c(K)$. So by Corollary 3.8, $\pi(\psi_n - \phi_n)v \rightarrow 0$ as $n \rightarrow \infty$. It follows that $\pi(\psi_n)v \rightarrow v$ as $n \rightarrow \infty$. But $\pi(\psi_n)v \in V^{\text{fin}}$. \square

Corollary 4.8. *$L^2(K)^{\text{fin}} \subset C(K)$ is a dense subspace. Moreover, if K is a Lie group then $L^2(K)^{\text{fin}} \subset C^k(K)$ is a dense subspace for $0 \leq k \leq \infty$.*

Proof. The claimed inclusions follow since matrix coefficients of finite dimensional representations of K are continuous, and moreover C^∞ in the case of Lie groups. The density then follows from Corollary 4.7. \square

Corollary 4.9. *If V is an irreducible continuous representation of K then V is finite dimensional.*

Proof. By Corollary 4.7, V^{fin} is dense in V . Hence $V^{\text{fin}} \neq 0$. Let ρ be a finite dimensional subrepresentation of V^{fin} . Then ρ is a closed invariant subspace of V . Hence $\rho = V$. \square

4.4. Smooth vectors. Let G be a Lie group. As we have noted in Subsection 1.2, any continuous *finite dimensional* representation $\pi : G \rightarrow \text{Aut}(V)$ is automatically smooth and thereby defines a representation $\pi_* : \mathfrak{g} \rightarrow \text{End}(V)$ of the corresponding Lie algebra, which determines π if G is connected. Moreover, if G is simply connected, this correspondence is an equivalence of categories. This immediately reduces the problem to pure algebra and is the main tool of studying finite dimensional representations of Lie groups.

We would like to have a similar theory for infinite dimensional representations. But in the infinite dimensional setting the above statements don't hold in the literal sense.

Example 4.10. Consider the action of S^1 on $L^2(S^1)$. Then the Lie algebra should act by $\frac{d}{d\theta}$. But this operator does not act on $L^2(S^1)$. The largest subspace of $L^2(S^1)$ preserved by this operator (acting on distributions on S^1) is $C^\infty(S^1)$.

This motivates the notion of a **smooth vector** in a continuous representation of a Lie group. To define this notion, for a manifold X and a topological vector space V , denote by $C^\infty(X, V)$ the space of smooth maps $X \rightarrow V$ (where smooth maps are defined in the same way as in the case of finite dimensional V).

Definition 4.11. Let (V, π) be a continuous representation of a Lie group G . A vector $v \in V$ is called **smooth** if the map $G \rightarrow V$ given by $g \mapsto \pi(g)v$ is smooth, i.e., belongs to $C^\infty(G, V)$. The space of smooth vectors is denoted by V^∞ .

It is clear that $V^\infty \subset V$ is a G -invariant subspace (although not a closed one).

Example 4.12. For the representation of a compact Lie group K on $V = L^2(K)$, we have $V^\infty = C^\infty(K)$.

Proposition 4.13. Let (V, π) be a continuous representation of a Lie group G with $\mathfrak{g} = \text{Lie}(G)$. Let $v \in V^\infty$. Then we have a linear map $\pi_{*,v} : \mathfrak{g} \rightarrow V^\infty$ given by

$$\pi_{*,v}(b) = \left. \frac{d}{dt} \right|_{t=0} \pi(e^{tb})v.$$

This defines a Lie algebra homomorphism $\pi_* : \mathfrak{g} \rightarrow \text{End}_{\mathbb{C}}(V^\infty)$ (algebra of all linear endomorphisms of V^∞) given by $\pi_*(b)v := \pi_{*,v}(b)v$.

Exercise 4.14. Prove Proposition 4.13.

Proposition 4.15. (i) V^∞ is dense in V .

(ii) $V^{\text{fin}} \subset V^\infty$.

Proof. (i) Let $\phi_n \rightarrow \delta_1$ be a smooth Dirac sequence. Then $\pi(\phi_n)v \rightarrow v$ as $n \rightarrow \infty$. But it is easy to see that $\pi(\phi_n)v \in V^\infty$.

(ii) This follows since matrix coefficients of finite dimensional representations are smooth. \square

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