11. **Proof of the CST theorem, part I**

11.1. **Proof of the CST theorem, part I, the "if" direction.** We first need a lemma from invariant theory. Let $G \subset GL(V)$ be a finite subgroup, and $I \subset \mathbb{C}[V]$ be the ideal generated by positive degree elements of $\mathbb{C}[V]^G$. Let $f_1, ..., f_r \in \mathbb{C}[V]^G$ be homogeneous generators of I (which exist by the Hilbert basis theorem).

Lemma 11.1. *The algebra $\mathbb{C}[V]^G$ is generated by $f_1, ..., f_r$; in particular, it is finitely generated.*

Proof. We need to show that every homogeneous $f \in \mathbb{C}[V]^G$ is a polynomial of $f_1, ..., f_r$. The proof is by induction in $d = \deg f$. The base $d = 0$ is obvious. If $d > 0$, we have $f \in I$, so

$$f = s_1 f_1 + ... + s_r f_r$$

where $s_i \in \mathbb{C}[V]$ are homogeneous of degrees $< d$.

For $h \in \mathbb{C}[V]$ let $h^* := \frac{1}{|G|}\sum_{g \in G} gh \in \mathbb{C}[V]^G$ be the G-average of h. Then we have

$$f = s_1^* f_1 + ... + s_r^* f_r.$$

But by the induction assumption, s_i^* are polynomials of $f_1, .., f_r$, which proves the lemma. □

Remark 11.2. Let A be a finitely generated commutative $\mathbb{C}$-algebra with an action of a finite group G. Lemma 11.1 implies that the algebra A^G is also finitely generated (the **Hilbert-Noether lemma**). Indeed, pick generators $a_1, ..., a_m$ of A and let $V \subset A$ be the (finite dimensional) G-submodule generated by them. Then A^G is a quotient of $(SV)^G = \mathbb{C}[V^*]^G$, which is finitely generated by Lemma 11.1.

The next lemma establishes a special property of algebras of invariants of complex reflection groups which will allow us to prove that they are polynomial algebras.

Lemma 11.3. *Assume that G is a complex reflection group. Let I be as above, $F_1, ..., F_m \in \mathbb{C}[V]^G$ be homogeneous, and suppose that F_1 does not belong to the ideal in $\mathbb{C}[V]^G$ generated by $F_2, ..., F_m$. Suppose $g_i \in \mathbb{C}[V]$ for $1 \le i \le m$ are homogeneous and $\sum_{i=1}^m g_i F_i = 0$. Then $g_1 \in I$.*

Proof. Let $J = (F_2, ..., F_m) \subset \mathbb{C}[V]$. We claim that $F_1 \notin J$. Indeed, if $F_1 = s_2 F_2 + ... + s_m F_m$ then $F_1 = s_2^* F_2 + ... + s_m^* F_m$, contradicting our assumption.

We prove the lemma by induction in $D := \deg g_1$. If $D = 0$ then $g_1 = 0$, as $F_1 \notin J$. This establishes the base of induction.

Now assume $D > 0$. Let $\sigma \in G$ be a complex reflection and α be the linear function on V defining the reflection hyperplane V^σ (i.e., the eigenvector of σ in V^* with eigenvalue $\neq 1$). Then $\sigma g_i - g_i$ vanishes on V^σ, so is divisible by α. Thus

$$\sigma g_i - g_i = h_i \alpha$$

for some polynomials h_i with $\deg h_i = \deg g_i - 1$, in particular $\deg h_1 = D - 1$. Applying the operator $\sigma - 1$ to the relation $\sum_{i=1}^m g_i F_i = 0$ and dividing by α, we obtain

$$\sum_{i=1}^m h_i F_i = 0.$$

By the induction assumption $h_1 \in I$, so $\sigma g_1 - g_1 \in I$. **Since W is generated by complex reflections,** this implies that $w g_1 - g_1 \in I$ for any $w \in G$. Thus $g_1^* - g_1 \in I$. But g_1^* is a positive degree invariant, so $g_1^* \in I$. Hence $g_1 \in I$, which justifies the induction step. □

Now we are ready to prove the "if" direction of the Chevalley-Shephard-Todd theorem. Suppose that $f_1, ..., f_r \in \mathbb{C}[V]^G$ are homogenous of positive degree and form a *minimal* set of homogeneous generators of I.

Lemma 11.4. *$f_1, ..., f_r$ are algebraically independent.*

Proof. Assume the contrary, i.e.,

$$h(f_1, ..., f_r) = 0, \tag{7}$$

where $h(y_1, ..., y_r)$ is a nonzero polynomial. Let $d_i := \deg f_i$. We may assume that h is quasi-homogeneous (with $\deg y_i = d_i$), of the lowest possible degree. Let x_k be linear coordinates on V, $\partial_k := \frac{\partial}{\partial x_k}$. Differentiating (7) with respect to x_k and using the chain rule, we get

$$\sum_{j=1}^r h_j(\mathbf{f}) \partial_k f_j = 0, \tag{8}$$

where $\mathbf{f} := (f_1, ..., f_r)$ and $h_j := \frac{\partial h}{\partial y_j}$. By renumbering f_j if needed, we may assume that $h_1(\mathbf{f}), ..., h_m(\mathbf{f})$ is a minimal generating set of the ideal $(h_1(\mathbf{f}), ..., h_r(\mathbf{f})) \subset \mathbb{C}[V]$. Moreover, since h is of lowest degree, $h_i(\mathbf{f}) \neq 0$ for some i, so $m \geq 1$. Then for $i > m$ we have

$$h_i(\mathbf{f}) = \sum_{j=1}^m g_{ij} h_j(\mathbf{f})$$

for some homogeneous polynomials $g_{ij} \in \mathbb{C}[V]$ of degree

$$\deg h_i - \deg h_j = d_j - d_i.$$

Substituting this into (8), we get

$$\sum_{j=1}^{m} p_j h_j(\mathbf{f}) = 0,$$

where

$$p_j := \partial_k f_j + \sum_{i=m+1}^{r} g_{ij} \partial_k f_i.$$

Since $h_1(\mathbf{f}) \notin (h_2(\mathbf{f}), ..., h_m(\mathbf{f}))$, by Lemma 11.3 applied to $F_i = h_i(\mathbf{f})$, $1 \le i \le m$, we have $p_1 \in I$. Thus

$$\partial_k f_1 + \sum_{i=m+1}^{r} g_{i1} \partial_k f_i = \sum_{i=1}^{r} q_{ik} f_i,$$

where $q_{ik} \in \mathbb{C}[V]$ are homogeneous of degree $d_1 - d_i - 1$. Let us multiply this equation by x_k and add over all k. Then we get

$$d_1 f_1 + \sum_{i=m+1}^{r} g_{i1} d_i f_i = \sum_{i=1}^{r} q_i f_i, \tag{9}$$

where $q_i := \sum_k x_k q_{ik}$. In particular, q_i are homogeneous of strictly positive degree. All terms in this equation are homogeneous of the same degree d_1, so we must have $q_1 = 0$. Thus (9) implies that $f_1 \in (f_2, ..., f_r)$, a contradiction with our minimality assumption. $\square$

Now, by Lemmas 11.4 and 11.1, we have $\mathbb{C}[V]^G = \mathbb{C}[f_1, ..., f_r]$. This proves the "if" direction of the Chevalley-Shephard-Todd theorem.

Remark 11.5. Note that $r = \mathrm{trdeg}(\mathbb{C}(V)^G) = \mathrm{trdeg}(\mathbb{C}(V)) = n$, where $n = \dim V$ and trdeg denotes the transcendence degree of a field, since transcendence degree does not change under finite extensions.

11.2. **A lemma on group actions.**

Lemma 11.6. *Let U be an affine space over $\mathbb{C}$ and G a finite group acting on U by polynomial automorphisms.*

(i) Let $u \in U$ be a point with trivial stabilizer in G. Then there exists a local coordinate system on U near u consisting of elements of $\mathbb{C}[U]^G$.

(ii) Maximal ideals in $\mathbb{C}[U]^G$ (i.e., characters $\chi : \mathbb{C}[U]^G \to \mathbb{C}$) are in bijection with G-orbits on U, which assigns to an orbit Gu the character $\chi_u(f) := f(u)$. Thus the set of maximal ideals in $\mathbb{C}[U]^G$ is U/G.

Proof. (i) Pick a basis $\{e_i\}$ of T_u^*U. Since $gu \neq u$ for any $g \in G$, $g \neq 1$, there exist $y_i \in \mathbb{C}[U]$, $1 \leq i \leq \dim U$ such that the linear approximation of y_i at gu is zero for all $g \neq 1$, $y_i(u) = 0$, and $dy_i(u) = e_i$. Let y_i^* be the average of y_i over G. Then $\{y_i^*\}$ form a required coordinate system.

(ii) Suppose $v, u \in U, v \notin Gu$, then $Gu \cap Gv = \emptyset$, so there exists $f \in \mathbb{C}[U]$ such that $f|_{Gv} = 0$, $f|_{Gu} = 1$. Moreover, by replacing f by f^*, we may choose such $f \in \mathbb{C}[U]^G$. Then $\chi_v(f) = 0$ while $\chi_u(f) = 1$, so $\chi_u \neq \chi_v$, hence $u \mapsto \chi_u$ is injective. To show that it's also surjective, take a maximal ideal $\mathfrak{m} \subset \mathbb{C}[U]^G$. It generates an ideal $I \subset \mathbb{C}[U]$ whose projection to $\mathbb{C}[U]^G$ is $\mathfrak{m}$. Thus I is a proper ideal, so by the Nullstellensatz, its zero set $Z \subset U$ is non-empty. Let $u \in Z$, then for any $f \in \mathfrak{m}$, $\chi_u(f) = f(u) = 0$. Hence $\mathfrak{m} = \mathrm{Ker}\chi_u$, as desired. □

11.3. **Proof of the CST theorem, part I, the "only if" direction.** [15]

Let $G \subset GL(V)$ be a finite subgroup. Let H be the normal subgroup of G generated by the complex reflections of G. Then by the "if" part of the theorem, $\mathbb{C}[V]^H$ is a polynomial algebra with an action of G/H. In other words, using Lemma 11.6(ii), $U := V/H$ is an affine space with a (possibly non-linear) action of G/H.

Moreover, we claim that G/H acts freely on U outside of a set of codimension ≥ 2. Indeed, if $1 \neq s \in G/H$ then hs is not a reflection for any $h \in H$, so $Y_s := \cup_{h\in H} V^{hs}$ has codimension ≥ 2. Now, for any v in the preimage of U^s in V we have $sv = h^{-1}v$ for some $h \in H$, thus $hsv = v$ and $v \in Y_s$. Thus U^s is contained in the image of Y_s in U, hence $\mathrm{codim}(U^s) \geq 2$, as claimed.

Now assume that $\mathbb{C}[V]^G$ is a polynomial algebra, and let $V/G = W$ be the corresponding affine space. Consider the natural regular map $\eta : V/H = U \to V/G = W$ between n-dimensional affine spaces, and let $J \in \mathbb{C}[U]$ be the Jacobian of this map (well defined up to scaling). If $u \in U$ and the stabilizer of u in G/H is trivial then by Lemma 11.6, η is étale at u, hence $J(u) \neq 0$. But as shown above, the complement of such points has codimension ≥ 2. This implies that $J = \mathrm{const}$, as a nonconstant polynomial would vanish on a subset of codimension 1. Thus by the inverse function theorem η is an isomorphism near 0, in particular bijective, hence $H = G$.

Remark 11.7. Let X be an smooth affine algebraic variety over $\mathbb{C}$ and G be a finite group of automorphisms of X. Then by the Hilbert-Noether lemma, $\mathbb{C}[X]^G$ is finitely generated, so $X/G := \mathrm{Spec}\mathbb{C}[X]^G$ is an affine algebraic variety. The Chevalley-Shephard-Todd theorem

[15] This proof uses some very basic algebraic geometry.

implies that X/G is smooth at the image $x^* \in X/G$ of $x \in X$ if and only if the stabilizer G_x of x is a complex reflection group in $GL(T_xX)$. In particular, X/G is smooth iff all stabilizers are complex reflection groups. This follows from the **formal Cartan lemma**: any action of a finite group G on a formal polydisk D over a field of characteristic zero is equivalent to its linearization (i.e., to the action of G on the formal neighborhood of 0 in the tangent space to D at its unique geometric point).

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