12. **Chevalley-Shephard-Todd theorem, part II**

12.1. **Degrees of a complex reflection group.** The degrees d_i of the generators f_i of $\mathbb{C}[V]^G$ for a complex reflection group G are uniquely determined up to relabelings (even though f_i themselves are not). Indeed, recall that for a $\mathbb{Z}$-graded vector space M with finite dimensional homogeneous components its **Hilbert series** is

$$H(M,q) = \sum_{i\in\mathbb{Z}} \dim M[i]q^i$$

(also called Hilbert polynomial if $\dim M < \infty$). Then the Hilbert series of $\mathbb{C}[V]^G$ is

$$H(\mathbb{C}[V]^G, q) = \frac{1}{\prod_{i=1}^r (1-q^{d_i})},$$

which uniquely determines d_i. These numbers are usually arranged in non-decreasing order and are called the **degrees** of G. For instance, for Weyl groups of classical simple Lie algebras we saw in Examples 10.3,10.4 that in type A_{n-1} the degrees are $2,3,...,n$, for B_n and C_n they are $2,4,...,2n$, and for D_n they are $2,4,...,2n-2$ and n. In particular, in the last case, if n is even, the degree n occurs twice.

12.2. $\mathbb{C}[V]$ **as a** $\mathbb{C}[V]^G$**-module.** Let R be a commutative ring. Let A be a commutative R-algebra with an R-linear action of a finite group G.

Proposition 12.1. (**Hilbert-Noether theorem**) *(i) A is integral over A^G. In particular, if A finitely generated then it is module-finite over A^G.*

(ii) If R is Noetherian and A is finitely generated then so is A^G.

Proof. (i) We will prove only the first statement, as the second one then follows immediately. For $a \in A$, consider the monic polynomial

$$P_a(x) := \prod_{g\in G}(x - ga).$$

It is easy to see that $P_a \in A^G[x]$ and $P_a(a) = 0$, which implies the statement.

(ii) This follows from (i) and the Artin-Tate lemma: If $B \subset A$ is an R-subalgebra of a finitely generated R-algebra A over a Noetherian ring R and A is module-finite over B then B is finitely generated.[16] □

[16] Recall the proof of the Artin-Tate lemma. Let $x_1,...,x_m$ generate A as an R-algebra and let $y_1,...,y_n$ generate A as a B-module. Then we can write

$$x_i = \sum_j b_{ij}y_j, \quad y_iy_j = \sum_k b_{ijk}y_k$$

This shows for any finite $G \subset GL(V)$, the algebra $\mathbb{C}[V]$ is module-finite over $\mathbb{C}[V]^G$. Note that in (ii) we again proved that $\mathbb{C}[V]^G$ is finitely generated.

Theorem 12.2. *(Chevalley-Shephard-Todd theorem, part II) If G is a complex reflection group then for any irreducible representation ρ of G, the $\mathbb{C}[V]^G$-module $\mathrm{Hom}_G(\rho, \mathbb{C}[V])$ is free of rank $\dim \rho$. Thus the G-module $R_0 = \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_n)$ is the regular representation and $\prod_{i=1}^n d_i = |G|$. Moreover, the Hilbert polynomial $H(R_0, q) := \sum_{N \geq 0} \dim R_0[N] q^N$ is*

$$H(R_0, q) = \prod_{i=1}^n [d_i]_q,$$

where $[d]_q := \frac{1-q^d}{1-q} = 1 + q + ... + q^{d-1}$.

Thus we see that the Hilbert polynomial of $\mathrm{Hom}_G(\rho, R_0)$ is some polynomial $K_\rho(q)$ with nonnegative integer coefficients and $K_\rho(1) = \dim \rho$. It is called the **Kostka polynomial**. We have

$$\sum_\rho K_\rho(q) \dim \rho = H(R_0, q) = \prod_{i=1}^n [d_i]_q.$$

For example, for $G = S_3$ and V the reflection representation we have three irreducible representations: $\mathbb{C}_+$ (trivial), $\mathbb{C}_-$ (sign) and V. We have $K_{\mathbb{C}_+}(q) = 1$ and

$$1 + 2K_V(q) + K_{\mathbb{C}_-}(q) = (1+q)(1+q^2) = 1 + 2q + 2q^2 + q^3.$$

It follows that

$$K_V(q) = q + q^2, \ K_{\mathbb{C}_-}(q) = q^3.$$

12.3. **Graded modules.** For the proof of Theorem 12.2 we need to recall some basics from commutative algebra, which we discuss in the next few subsections.

Let k be a field, S a $\mathbb{Z}_+$-graded (not necessarily commutative) k-algebra with generators f_i of positive integer degrees $\deg f_i = d_i$, M a $\mathbb{Z}_+$-graded left S-module, and $M_0 := M/S_+M$, where $S_+ \subset S$ is the augmentation ideal.

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with $b_{ij}, b_{ijk} \in B$. Then A is module-finite over the R-algebra $B_0 \subset B$ generated by b_{ij}, b_{ijk} (namely, it is generated as a module over B_0 by the y_i). Using that R and hence B_0 is Noetherian, we obtain that B is also module-finite over B_0. Since B_0 is a finitely generated R-algebra, so is B.

Lemma 12.3. *(i) Any homogeneous lift $\{v_i^*\}$ of a homogeneous basis $\{v_i\}$ of M_0 to M is a system of generators for M; in particular, if* $\dim M_0 < \infty$ *then M is finitely generated.*

(ii) If in addition M is projective, then $\{v_i^*\}$ is actually a basis of M (in particular, M is free). Thus if $\dim M_0[i] < \infty$ *for all i then*

$$H(M,q) = H(M_0,q)H(S,q).$$

In particular, if $S = k[f_1, ..., f_n]$ then

$$H(M,q) = \frac{H(M_0,q)}{\prod_{i=1}^n (1-q^{d_i})}.$$

Proof. (i) We prove that any homogeneous element $u \in M$ is a linear combination of v_i^* with coefficients in S by induction in $\deg u$ (with obvious base). Namely, if u_0 is the image of u in M_0 then $u_0 = \sum_i c_i v_i$ for some $c_i \in k$ ($c_i = 0$ unless $\deg v_i = \deg u$), and so

$$u - \sum_i c_i v_i^* = \sum_j f_j u_j,$$

with $\deg u_j = \deg u - d_j$. So by the induction assumption $u_j = \sum_i p_{ij} v_i^*$ for some homogeneous $p_{ij} \in S$ of degree $\deg u - d_j - \deg v_i^*$, and we get

$$u = \sum_i p_i v_i^*,$$

where $p_i := c_i + \sum_j f_j p_{ij}$.[17]

(ii) Let M' be the free graded S-module with basis w_i of degrees $\deg w_i = \deg v_i$, and $f : M' \to M$ be the surjection sending w_i to v_i^*. Since M is projective, the map

$$f\circ : \mathrm{Hom}(M, M') \to \mathrm{Hom}(M, M)$$

is surjective, so we can pick a homogeneous $g : M \to M'$ of degree 0 such that $f \circ g = \mathrm{id}_M$. Then $g \circ f : M' \to M'$ is a projection which identifies M' with $M \oplus \mathrm{Ker} f$ as a graded S-module. But the map $f_0 : M'_0 \to M_0$ induced by f sends the basis w_i of M'_0 to the basis v_i of M_0, so is an isomorphism. It follows that $(\mathrm{Ker} f)_0 = 0$, so $\mathrm{Ker} f = 0$ and f is an isomorphism, as claimed. ☐

12.4. **Koszul complexes.** Let R be a commutative ring and $f \in R$. Then we can define a 2-step **Koszul complex** $K_R(f) = [R \to R]$ with the differential given by multiplication by f (the two copies of R sit in degrees -1 and 0). We have $H^0(K_R(f)) = R/(f)$, and $K_R(f)$ is exact

[17] Note that for each i, one of these two summands is necessarily 0.

in degree -1 if and only if f is not a zero divisor in R. This allows us to define the Koszul complex of several elements of R:

$$K_R(f_1, ..., f_m) = K_R(f_1) \otimes_R ... \otimes_R K_R(f_m)$$

with $H^0(K_R(f_1, ..., f_m)) = R/(f_1, ..., f_m)$. Thus

$$K_R(f_1, ..., f_m) = K_R(f_1, ..., f_{m-1}) \otimes_R K_R(f_m).$$

For example, let $R := k[x_1, ..., x_n]$ for a field k. Then the complex $K_n := K_R(x_1, ..., x_n) = K_1^{\otimes n}$ is acyclic in negative degrees and has $H^0 = k$. Thus for any commutative k-algebra S, the complex $K_{R\otimes S}(x_1, ..., x_n) := K_R(x_1, ..., x_n) \otimes S$ is acyclic in negative degrees and has $H^0 = S$. By taking $S = R$ and making a linear change of variable, this yields a free resolution of R as an R-bimodule called the **Koszul resolution**, which we'll denote it by K_n:

$$0 \to R \otimes \wedge^n k^n \otimes R \to ... \to R \otimes \wedge^2 k^n \otimes R \to R \otimes k^n \otimes R \to R \otimes R \to R.$$

Now if M is any R-module then $K_n \otimes_R M$ is a free resolution of M of length n. Thus we obtain

Proposition 12.4. *If $i > n$ then for any $k[x_1, ..., x_n]$-modules M, N, one has* $\mathrm{Ext}^i(M, N) = 0$.

12.5. **Syzygies.** Now assume that M is a finitely generated graded module over $R = k[x_1, ..., x_n]$. Then $M =: M_0$ is a quotient of $R \otimes V_0$, where V_0 is a finite dimensional graded vector space. By the Hilbert basis theorem, the kernel M_1 of the map $\phi_0 : R \otimes V_0 \to M$ is finitely generated, so is a quotient of $R \otimes V_1$ for some finite dimensional graded space V_1, and the kernel M_2 of $\phi_1 : R \otimes V_1 \to M_1$ is finitely generated, and so on. The long exact sequences of Ext groups associated to the short exact sequences

$$0 \to M_{j+1} \to R \otimes V_j \to M_j \to 0$$

and Proposition 12.4 then imply by induction in j that $\mathrm{Ext}^i(M_j, N) = 0$ for any R-module N if $i > n - j$. In particular, the module M_n is projective, hence free by Lemma 12.3, i.e., we may take V_n such that $M_n = R \otimes V_n$. This gives a free resolution of M by finitely generated graded R-modules:

$$0 \to R \otimes V_n \to ... \to R \otimes V_0 \to M.$$

Thus, taking graded Euler characteristic we obtain

Theorem 12.5. *(***Hilbert syzygies theorem***) We have*

$$H(M, q) = \frac{p(q)}{(1-q)^n},$$

where p is a polynomial with integer coefficients.

Proof. Indeed, p is just the alternating sum of the Hilbert polynomials of V_j. $\square$

12.6. **The Hilbert-Samuel polynomial.** Let R be a commutative Noetherian ring and $\mathfrak{m} \subset R$ a maximal ideal. Then $R/\mathfrak{m} = k$ is a field and $\mathfrak{m}^N/\mathfrak{m}^{N+1}$ is a finite dimensional k-vector space. Thus $\mathrm{gr}(R) := \oplus_{N\ge 0}\mathfrak{m}^N/\mathfrak{m}^{N+1}$ (where $\mathfrak{m}^0 := R$) is a graded algebra generated in degree 1. So by the Theorem 12.5, the Hilbert series

$$H(\mathrm{Gr}(R), q) = \sum_{N\ge 0} \dim_k(\mathfrak{m}^N/\mathfrak{m}^{N+1})q^N$$

is a rational function of the form $\frac{p(q)}{(1-q)^m}$, where p is a polynomial and $m = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. Hence

$$P_{R,\mathfrak{m}}(N) := \sum_{j=0}^{N-1} \dim_k(\mathfrak{m}^j/\mathfrak{m}^{j+1}) = \mathrm{length}(R/\mathfrak{m}^N)$$

is a polynomial in N for large enough N called the **Hilbert-Samuel polynomial** of R at $\mathfrak{m}$. The degree of this polynomial equals the order of the pole of $H(\mathrm{Gr}(R), q)$ at $q = 1$. We call this degree the **dimension** of R at $\mathfrak{m}$, denoted $\dim_{\mathfrak{m}} R$. For example, if $R = k[x_1, ..., x_n]$ and $\mathfrak{m}$ is any maximal ideal then $P_{R,\mathfrak{m}}(N) = \binom{N+n-1}{n}$, so $\dim_{\mathfrak{m}} R = n$.

Lemma 12.6. *Let $f \in \mathfrak{m}$. Then* $\dim_{\mathfrak{m}}(R/f) \ge \dim_{\mathfrak{m}} R - 1$.

Proof. The ideal (f) in $R/\mathfrak{m}^N$ is the image of $fR/\mathfrak{m}^{N-1}$. So we have

$$P_{R/f,\mathfrak{m}}(N) = \mathrm{length}((R/\mathfrak{m}^N)/f) \ge \mathrm{length}(R/\mathfrak{m}^N) - \mathrm{length}(R/\mathfrak{m}^{N-1})$$
$$= P_{R,\mathfrak{m}}(N) - P_{R,\mathfrak{m}}(N-1),$$

which implies the statement. $\square$

Let k be an algebraically closed field and $\mathfrak{m}_p \subset k[x_1, ..., x_n]$ be the maximal ideal corresponding to $p \in k^n$.

Corollary 12.7. *Let $f_1, ..., f_m \in k[x_1, ..., x_n]$ be homogeneous polynomials. Let Z be an irreducible component of the zero set $Z(f_1, ..., f_m) \subset k^n$. Then* $\dim_{\mathfrak{m}_0} k[Z] \ge n - m$.

Proof. Let $p \in Z$ be not contained in other components of $Z(f_1, ..., f_m)$. Applying Lemma 12.6 repeatedly, we get $\dim_{\mathfrak{m}_p} k[Z] \ge n - m$. But $\mathfrak{m}_0 = \mathrm{gr}(\mathfrak{m}_p)$, hence $\mathfrak{m}_0^N \subset \mathrm{gr}(\mathfrak{m}_p^N)$ and $k[Z]/\mathfrak{m}_p^N$ is a quotient of $k[Z]/\mathfrak{m}_0^N$. Thus $\dim_{\mathfrak{m}_0} k[Z] \ge \dim_{\mathfrak{m}_p} k[Z]$[18], so $\dim_{\mathfrak{m}_0} k[Z] \ge n-m$. $\square$

[18]In fact these dimensions are equal (to $\dim Z$), but we don't use it here.

12.7. **Regular sequences.** Let R be a commutative ring. A sequence $f_1, ..., f_n \in R$ is called a **regular sequence** if for each $j \in [1, n]$, f_j is not a zero divisor in $R/(f_1, ..., f_{j-1})$, and $R/(f_1, ..., f_n) \neq 0$.

Lemma 12.8. *If $f_1, ..., f_n \in R$ is a regular sequence then the complex $K_R(f_1, ..., f_n)$ is exact in negative degrees.*

Proof. The proof is by induction in n with obvious base. For the induction step, note that by the inductive assumption $K_R(f_1, ..., f_{n-1})$ is exact in negative degrees with $H^0 = R/(f_1, ..., f_{n-1})$, so the cohomology of $K_R(f_1, ..., f_n)$ coincides with the cohomology $K_{R/(f_1,...,f_{n-1})}(f_n)$, which vanishes in negative degrees since f_n is not a zero divisor in $R/(f_1, ..., f_{n-1})$. $\square$

Now let k be an algebraically closed field.

Proposition 12.9. *Suppose $f_1, ..., f_n \in R := k[x_1, ..., x_n]$ are homogeneous polynomials of positive degree such that the zero set $Z(f_1, ..., f_n)$ consists of the origin. Then $f_1, ..., f_n$ is a regular sequence.*

Proof. We need to show that for each $m \le n - 1$, f_{m+1} is not a zero divisor in $R_m := k[x_1, ..., x_n]/(f_1, ..., f_m)$. Let $Z_m = Z(f_1, ..., f_m)$. It suffices to show that f_{m+1} does not vanish on any irreducible component of Z_m. Assume the contrary, i.e., that it vanishes on such a component Z_m^0. By Corollary 12.7, we have $\dim_{\mathfrak{m}_0} k[Z_m^0] \ge n - m$. Since $f_{m+1} = 0$ on Z_m^0, using Lemma 12.6 repeatedly, we get

$$\dim_{\mathfrak{m}_0} k[Z_m^0]/(f_{m+1}, ..., f_n) \ge 1,$$

which is a contradiction, as the zero set of $f_{m+1}, ..., f_n$ on Z_m^0 consists just of the origin, so this dimension must be zero. $\square$

Proposition 12.10. *Suppose $f_1, .., f_n \in R := k[x_1, ..., x_n]$ are homogeneous polynomials of degrees $d_1, ..., d_n > 0$ such that R is a finitely generated module over $S := k[f_1, ..., f_n]$. Then this module is free of rank $\prod_{i=1}^n d_i$. Moreover, the Hilbert polynomial of $R_0 := k[x_1, ..., x_n]/(f_1, ..., f_m)$ (or, equivalently, of a space of free homogeneous generators of this module) is*

$$H(R_0, q) = \prod_{i=1}^{n} [d_i]_q. \tag{10}$$

Proof. By Lemma 12.3, it suffices to show that R is a free S-module. By assumption R_0 is finite dimensional, i.e., the equations

$$f_1 = ... = f_n = 0$$

have only the zero solution. By Proposition 12.9, this implies that $f_1, ..., f_n$ is a regular sequence, so by Lemma 12.8 the Koszul complex

$K_R(f_1, ..., f_n)$ associated to this sequence is exact in negative degrees. Now, write S as $k[a_1, ..., a_n]$ with $\deg a_j = 0$ and consider the complex $K_{R\otimes S}(f_1 - a_1, ..., f_n - a_n)$. This complex is filtered by degree with associated graded being

$$K_{R\otimes S}(f_1, ..., f_n) = K_R(f_1, ..., f_n) \otimes S.$$

Thus $K_{R\otimes S}(f_1 - a_1, ..., f_n - a_n)$ is also exact in nonzero degrees with

$$H^0 = k[x_1, ..., x_n, a_1, ..., a_n]/(f_1 - a_1, ..., f_n - a_n) = R.$$

and the associated graded under the above filtration is $\mathrm{gr}(R) = R_0 \otimes S$ as an S-module. This module is free over S, hence so is R. $\square$

Remark 12.11. Let $f_1, ..., f_r$ be a regular sequence of homogeneous polynomials in $k[x_1, ..., x_n]$ of positive degree and $Z_m \subset k^n$ be the zero set of $f_1, ..., f_m$. Then f_{m+1} is not a zero divisor in $k[x_1, ..., x_n]/(f_1, ..., f_m)$, hence does not vanish identically on any irreducible component of Z_m. So by induction in m we get that the dimension of every irreducible component of Z_m is $\le n-m$. By Corollary 12.7, this implies that this dimension is precisely $n-m$; in particular, $r \le n$, and every irreducible component of the affine scheme $\mathcal{Z} := \mathrm{Spec} k[x_1, ..., x_n]/(f_1, ..., f_r)$ has dimension $n - r$. Such a scheme is called a **complete intersection**. In fact, it follows by induction in r that $\mathcal{Z}$ is a complete intersection precisely when all its irreducible components have dimension $\le n - r$ (in which case they have dimension exactly $n - r$). In particular, if $r = n$, this means that the only k-point of $\mathcal{Z}$ is the origin, as indicated in Proposition 12.9. Thus the converse of this proposition also holds.

12.8. **Proof of the CST Theorem, Part II.** We are now ready to prove Theorem 12.2. It follows from Proposition 12.10, Lemma 12.1 and Theorem 10.6 that $\mathbb{C}[V]$ is a free $\mathbb{C}[V]^G$-module. Since $\mathbb{C}[V] = \oplus_\rho \mathrm{Hom}_G(\rho, \mathbb{C}[V]) \otimes \rho$, it follows by Lemma 12.3(ii) that $\mathrm{Hom}_G(\rho, \mathbb{C}[V])$ is also a free $\mathbb{C}[V]^G$-module (as it is graded and projective). Finally, the rank of this module equals

$$\dim_{\mathbb{C}(V)^G}(\mathbb{C}(V)^G \otimes_{\mathbb{C}[V]^G} \mathrm{Hom}_G(\rho, \mathbb{C}[V])) = \dim_{\mathbb{C}(V)^G} \mathrm{Hom}_G(\rho, \mathbb{C}(V)),$$

which equals $\dim \rho$ by basic Galois theory ($\mathbb{C}(V)$ is a regular representation of G over $\mathbb{C}(V)^G$).