13. **Kostant's theorem**

13.1. **Kostant's theorem for $S\mathfrak{g}$.** Let $\mathfrak{g}$ be a semisimple complex Lie algebra.

Theorem 13.1. *(Kostant) $S\mathfrak{g}$ is a free $(S\mathfrak{g})^{\mathfrak{g}}$-module. Moreover, for every finite dimensional irreducible representation V of $\mathfrak{g}$, the space* $\mathrm{Hom}_{\mathfrak{g}}(V, S\mathfrak{g})$ *is a free $(S\mathfrak{g})^{\mathfrak{g}}$ module of rank* $\dim V[0]$, *the dimension of the zero weight space of V.*

The rest of the subsection is dedicated to the proof of this theorem. Introduce a filtration on $S\mathfrak{g}$ by setting $\deg(\mathfrak{g}_\alpha) = 1$ for all roots α and $\deg \mathfrak{h} = 2$. Then $\mathrm{gr}(S\mathfrak{g}) = S\mathfrak{n}_- \otimes S\mathfrak{h} \otimes S\mathfrak{n}_+$ and by the Chevalley restriction theorem, $\mathrm{gr}(S\mathfrak{g})^{\mathfrak{g}}$ is identified with the subalgebra $(S\mathfrak{h})^W$ of the middle factor. Thus by the Chevalley-Shephard-Todd theorem, $\mathrm{gr}(S\mathfrak{g})$ is a free $\mathrm{gr}(S\mathfrak{g})^{\mathfrak{g}}$-module. It follows that $S\mathfrak{g}$ is a free $(S\mathfrak{g})^{\mathfrak{g}}$-module (namely, any lift of a homogeneous basis of the graded module is a basis of the filtered module).

Now recall that

$$S\mathfrak{g} = \oplus_{V \in \mathrm{Irr}(\mathfrak{g})} V \otimes \mathrm{Hom}_{\mathfrak{g}}(V, S\mathfrak{g}). \tag{11}$$

Thus $\mathrm{Hom}_{\mathfrak{g}}(V, S\mathfrak{g})$ is a graded direct summand in $S\mathfrak{g}$. It follows that $\mathrm{Hom}_{\mathfrak{g}}(V, S\mathfrak{g})$ is a projective, hence free $(S\mathfrak{g})^{\mathfrak{g}}$-module (using Lemma 12.3(ii)).

It remains to prove the formula for the rank of $\mathrm{Hom}_{\mathfrak{g}}(V, S\mathfrak{g})$. To this end, consider the Q-graded Hilbert series of $S\mathfrak{g}$, i.e., the generating function of the characters of symmetric powers of $\mathfrak{g}$:

$$H_Q(S\mathfrak{g}, q) := \sum_{m \ge 0} \Big(\sum_{\mu \in Q} \dim S^m\mathfrak{g}[\mu] e^\mu\Big) q^m \in \mathbb{C}[Q][[q]].$$

Since $S\mathfrak{g} = S\mathfrak{h} \otimes \bigotimes_{\alpha \in R} S\mathfrak{g}_\alpha$, we have

$$H_Q(S\mathfrak{g}, q) = \frac{1}{(1-q)^r} \prod_{\alpha \in R} \frac{1}{1 - q e^\alpha},$$

where $r = \mathrm{rank}(\mathfrak{g})$. On the other hand, by (11),

$$H_Q(S\mathfrak{g}, q) = \sum_{V \in \mathrm{Irr}(\mathfrak{g})} H(\mathrm{Hom}_{\mathfrak{g}}(V, S\mathfrak{g}), q) \chi_V,$$

where χ_V is the character of V.

Now, by the Chevalley restriction theorem $(S\mathfrak{g})^{\mathfrak{g}} \cong (S\mathfrak{h})^W$, so

$$H(\mathrm{Hom}_{\mathfrak{g}}(V, S\mathfrak{g}), q) = H(\mathrm{Hom}_{\mathfrak{g}}(V, (S\mathfrak{g})_0), q) H((S\mathfrak{h})^W, q).$$

Thus by the Chevalley-Shephard-Todd theorem,

$$H(\mathrm{Hom}_{\mathfrak{g}}(V, S\mathfrak{g}), q) = H(\mathrm{Hom}_{\mathfrak{g}}(V, (S\mathfrak{g})_0), q) \prod_{i=1}^{r} \frac{1}{1-q^{d_i}}.$$

So we get

$$\sum_{V \in \mathrm{Irr}(\mathfrak{g})} H(\mathrm{Hom}_{\mathfrak{g}}(V, (S\mathfrak{g})_0), q)\chi_V = \frac{\prod_{i=1}^{r}[d_i]_q}{\prod_{\alpha\in R}(1-qe^{\alpha})}.$$

By character orthogonality, $H(\mathrm{Hom}_{\mathfrak{g}}(V, (S\mathfrak{g})_0), q)$ is the inner product of the right hand side of this equality with χ_V:

$$H(\mathrm{Hom}_{\mathfrak{g}}(V, (S\mathfrak{g})_0), q) = \left(\frac{\prod_{i=1}^{r}[d_i]_q}{\prod_{\alpha\in R}(1-qe^{\alpha})}, \chi_V\right).$$

Recall that the inner product on $\mathbb{C}[P]$ making the characters orthonormal is given by the formula

$$(\phi, \psi) = \frac{1}{|W|}\mathrm{CT}(\phi\psi^* \prod_{\alpha\in R}(1-e^{\alpha})),$$

where where CT denotes the constant term and $*$ is the automorphism of $\mathbb{C}[P]$ given by $(e^{\mu})^* = e^{-\mu}$. Thus, using that $\chi_V^* = \chi_{V^*}$, we get

$$H(\mathrm{Hom}_{\mathfrak{g}}(V, (S\mathfrak{g})_0), q) = \frac{\prod_{i=1}^{r}[d_i]_q}{|W|}\mathrm{CT}\left(\chi_{V^*}\prod_{\alpha\in R}\frac{1-e^{\alpha}}{1-qe^{\alpha}}\cdot\right). \tag{12}$$

In this formula q is a formal parameter, but the right hand side converges to an analytic function in the disk $|q| < 1$, since it can be written as an integral:

$$H(\mathrm{Hom}_{\mathfrak{g}}(V, (S\mathfrak{g})_0), q) = \frac{\prod_{i=1}^{r}[d_i]_q}{|W|}\int_{\mathfrak{h}_{\mathbb{R}}/Q^{\vee}} \chi_{V^*}(e^{ix})\prod_{\alpha\in R}\frac{1-e^{i\alpha(x)}}{1-qe^{i\alpha(x)}}dx,$$

where $Q^{\vee}$ is the coroot lattice. If $0 \le q < 1$, this can also be written as

(13)

$$H(\mathrm{Hom}_{\mathfrak{g}}(V, (S\mathfrak{g})_0), q) = \frac{\prod_{i=1}^{r}[d_i]_q}{|W|}\int_{\mathfrak{h}_{\mathbb{R}}/Q^{\vee}} \chi_{V^*}(e^{ix})\left|\prod_{\alpha\in R_+}\frac{1-e^{i\alpha(x)}}{1-qe^{i\alpha(x)}}\right|^2 dx.$$

Lemma 13.2. *As $q \to 1$ in $(0, 1)$, the function $F_q(x) := \prod_{\alpha\in R_+}\frac{1-e^{i\alpha(x)}}{1-qe^{i\alpha(x)}}$ goes to 1 in $L^2(\mathfrak{h}/Q^{\vee})$.*[19]

[19]Note however that $F_q(x)$ does not go to 1 pointwise (hence not in $C(\mathfrak{h}/Q^{\vee})$) since $F_q(0) = 0$.

Proof. If $x \in \mathbb{R}$, $|x| \le 1$ then $\min_{q\in[0,1]}(1-2qx+q^2)$ is 1 if $x \le 0$ and $1-x^2$ if $x > 0$. So if $z = x+iy$ is on the unit circle and $0 \le q < 1$ then

$$\left|\frac{1-z}{1-qz}\right|^2 = \frac{2(1-x)}{1-2qx+q^2} \le \begin{cases} 2(1-x), & x \le 0 \\ \frac{2}{1+x}, & x > 0 \end{cases} \le 4.$$

Note also that by the residue formula

$$\int_0^1 \frac{dt}{|1-qe^{2\pi it}|^2} = \frac{1}{2\pi i}\int_{|z|=1} \frac{z^{-1}dz}{(1-qz)(1-qz^{-1})} = \frac{1}{1-q^2}.$$

Thus

$$\int_0^1 \left|\frac{1-e^{2\pi it}}{1-qe^{2\pi it}} - 1\right|^2 dt = \int_0^1 \left|\frac{(q-1)e^{2\pi it}}{1-qe^{2\pi it}}\right|^2 dt = \frac{1-q}{1+q}.$$

So $\frac{1-z}{1-qz} \to 1$ as $q \to 1$ in $L^2(S^1)$. But if X is a finite measure space and for $j = 1, ..., N$, $f_n^{(j)} \to f^{(j)}$ in $L^2(X)$ as $n \to \infty$ and $|f_n^{(j)}(z)| \le C$ for all $n, j, z \in X$ then $\prod_j f_n^{(j)} \to \prod_j f_j$ in $L^2(X)$. This implies the statement. $\square$

By Lemma 13.2 we may take the limit $q \to 1$ under the integral in (13). Then, using that $\prod_{i=1}^r d_i = |W|$, we get

$$\dim \mathrm{Hom}_{\mathfrak{g}}(V, (S\mathfrak{g})_0) = \int_{\mathfrak{h}/Q^\vee} \chi_{V^*}(e^{ix})dx =$$

$$\mathrm{CT}(\chi_{V^*}) = \dim V^*[0] = \dim V[0],$$

which concludes the proof of Kostant's theorem.

13.2. **The structure of $S\mathfrak{g}$ as a $(S\mathfrak{g})^{\mathfrak{g}}$-module.** As a by-product, we obtain

Theorem 13.3. *(Kostant) For $\lambda \in P_+$ we have*

$$H(\mathrm{Hom}_{\mathfrak{g}}(L_\lambda^*, (S\mathfrak{g})_0), q) = \frac{\prod_{i=1}^r [d_i]_q}{|W|}\mathrm{CT}\left(\prod_{\alpha\in R}\frac{1-e^\alpha}{1-qe^\alpha}\chi_{L_\lambda}\right) =$$

$$\prod_{i=1}^r [d_i]_q \cdot \mathrm{CT}\left(\frac{e^\lambda \prod_{\alpha\in R_+}(1-e^\alpha)}{\prod_{\alpha\in R}(1-qe^\alpha)}\right).$$

Indeed, the first expression is (12) and second expression is obtained from (12) using the Weyl character formula for χ_{L_λ} and observing that all terms in the resulting sum over W are the same.

Substituting $\lambda = 0$, we get

Corollary 13.4.
$$\frac{1}{|W|}\mathrm{CT}\left(\prod_{\alpha\in R}\frac{1-e^\alpha}{1-qe^\alpha}\right)=\mathrm{CT}\left(\frac{\prod_{\alpha\in R_+}(1-e^\alpha)}{\prod_{\alpha\in R}(1-qe^\alpha)}\right)=\frac{1}{\prod_{i=1}^r [d_i]_q}.$$

For example, if $\mathfrak{g}=\mathfrak{sl}_2$, this formula looks like
(14)
$$\frac{1}{2}\mathrm{CT}\left(\frac{(1-z)(1-z^{-1})}{(1-qz)(1-qz^{-1})}\right)=\mathrm{CT}\left(\frac{1-z}{(1-qz)(1-qz^{-1})}\right)=\frac{1}{1+q},$$
which is easy to check using the residue formula.

For $\mathfrak{g}=\mathfrak{sl}_n$ we obtain the identity
$$\frac{1}{n!}\mathrm{CT}\left(\prod_{1\le i<j\le n}\frac{(1-\frac{X_i}{X_j})(1-\frac{X_j}{X_i})}{(1-q\frac{X_i}{X_j})(1-q\frac{X_j}{X_i})}\right)=\mathrm{CT}\left(\prod_{1\le i<j\le n}\frac{1-\frac{X_i}{X_j}}{(1-q\frac{X_i}{X_j})(1-q\frac{X_j}{X_i})}\right)$$
$$=\frac{1}{(1+q)...(1+q+...+q^{n-1})}.$$

13.3. **The structure of $U(\mathfrak{g})$ as a $Z(\mathfrak{g})$-module.** Recall that the universal enveloping algebra $U(\mathfrak{g})$ of any Lie algebra $\mathfrak{g}$ has the standard filtration defined on generators by $\deg(\mathfrak{g})=1$, which is called the **Poincaré-Birkhoff-Witt filtration**.

Let $\mathfrak{g}$ be a semisimple complex Lie algebra of rank r, and W be the Weyl group of $\mathfrak{g}$ with degrees $d_i, i=1,...,r$.

Theorem 13.5. *(Kostant) (i) The center $Z(\mathfrak{g})=U(\mathfrak{g})^{\mathfrak{g}}$ of $U(\mathfrak{g})$ is a polynomial algebra in r generators C_i of Poincaré-Birkhoff-Witt filtration degrees d_i.*

(ii) $U(\mathfrak{g})$ is a free module over $Z(\mathfrak{g})$, and for every irreducible finite dimensional representation V of $\mathfrak{g}$, the space $\mathrm{Hom}_{\mathfrak{g}}(V,U(\mathfrak{g}))$ is a free $Z(\mathfrak{g})$-module of rank $\dim V[0]$.

Proof. By the Poincaré-Birkhoff-Witt theorem, for any Lie algebra $\mathfrak{g}$ we have $\mathrm{gr}(U(\mathfrak{g}))=S\mathfrak{g}$. Moreover, we have the symmetrization map $S\mathfrak{g}\to U(\mathfrak{g})$ given by
$$a_1\otimes...\otimes a_n\mapsto\frac{1}{n!}\sum_{s\in S_n}a_{s(1)}...a_{s(n)},$$
$a_i\in\mathfrak{g}$, which is an isomorphism of $\mathfrak{g}$-modules. Using this map, any homogeneous element of $(S\mathfrak{g})^{\mathfrak{g}}$ can be lifted into $U(\mathfrak{g})^{\mathfrak{g}}$. It follows that $\mathrm{gr}(U(\mathfrak{g})^{\mathfrak{g}})=(S\mathfrak{g})^{\mathfrak{g}}$. Thus Theorem 13.1 implies all the statements of the theorem. □

Example 13.6. Suppose $\mathfrak{g}$ is simple. Then $d_1=2$ and C_1 is the quadratic Casimir of $\mathfrak{g}$.

Exercise 13.7. Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ spanned by elementary matrices E_{ij} with $\sum_{i=1}^n E_{ii} = 0$.

(i) Show that the center $Z(\mathfrak{g})$ is freely generated by the elements

$$C_{k-1} := \sum_{i_1,...,i_k=1}^{n} \prod_{j=1}^{k} E_{i_j,i_{j+1}},\ \ k = 2, ..., n.$$

where j is viewed as an element of $\mathbb{Z}/k$.

Hint: It is slightly more convenient (and equivalent) to consider $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, in which case one also has the generator C_0. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using the trace pairing on $\mathfrak{g}$. Let $T_k : \mathfrak{g}^{\otimes k} \to \mathbb{C}$ be the $\mathfrak{g}$-module map defined by $T_k(a_1 \otimes ... \otimes a_k) := \mathrm{Tr}(a_k...a_1)$. Let $T_k^* : \mathbb{C} \to \mathfrak{g}^{\otimes k}$ be the dual map. Show that

$$T_k^*(1) = \sum_{i_1,...,i_k=1}^{n} E_{i_1 i_2} \otimes E_{i_2 i_3} \otimes ... \otimes E_{i_k i_1}.$$

Use that this element is $\mathfrak{g}$-invariant to show that the element C_{k-1} is central.

(ii) Generalize these statements to $\mathfrak{so}_{2n+1}(\mathbb{C})$ and $\mathfrak{sp}_{2n}(\mathbb{C})$. What happens for $\mathfrak{so}_{2n}$?