15. **Category $\mathcal{O}$ of $\mathfrak{g}$-modules - I**

15.1. **Category $\mathcal{O}$.** Let $\mathfrak{g}$ be a semisimple complex Lie algebra.

Definition 15.1. The category $\mathcal{O} = \mathcal{O}_{\mathfrak{g}}$ is the full subcategory of $\mathfrak{g}$-mod, which consists of finitely generated $\mathfrak{g}$-modules M with weight decomposition and $P(M) \subset \cup_{i=1}^{m}(\lambda_i - Q_+)$, where $\lambda_1, ..., \lambda_m \in \mathfrak{h}^*$.

It is clear that $\mathcal{O}$ is closed under taking subquotients and direct sums, so it is an abelian category (recall that a submodule of a finitely generated $\mathfrak{g}$-module is finitely generated since $U(\mathfrak{g})$ is Noetherian).

Also it is easy to see that any nonzero object $M \in \mathcal{O}$ has a singular vector (namely, take any nonzero vector of a maximal weight in $P(M)$). Thus the simple objects (=modules) of $\mathcal{O}$ are L_λ, $\lambda \in \mathfrak{h}^*$.

Example 15.2. All highest weight $\mathfrak{g}$-modules, in particular a Verma module M_λ and its simple quotient L_λ belong to $\mathcal{O}$. Another example is $\overline{M}_{-\lambda}^*$, the restricted dual to the lowest weight Verma module $\overline{M}_{-\lambda}$, introduced in Exercise 8.13(ii). This module is called the **contragredient Verma module** and denoted $M_\lambda^\vee$.

Lemma 15.3. *If $M \in \mathcal{O}$ then the weight subspaces of M are finite dimensional.*

Proof. Let $v_1, .., v_m$ be generators of M which are eigenvectors of $\mathfrak{h}$ (they exist since M is finitely generated and has weight decomposition). Let $E := \sum_{i=1}^m U(\mathfrak{h} \oplus \mathfrak{n}_+)v_i = \sum_{i=1}^m U(\mathfrak{n}_+)v_i$. Then E is finite dimensional by the condition on the weights of M. On the other hand, the natural map $U(\mathfrak{n}_-) \otimes E \to M$ is surjective. The lemma follows, as weight subspaces of $U(\mathfrak{n}_-) \otimes E$ are finite dimensional. □

Let $\mathcal{R}$ be the ring of series $F := \sum_{\mu \in \mathfrak{h}^*} c_\mu e^\mu$, where $c_\mu \in \mathbb{Z}$ and the set $P(F)$ of μ with $c_\mu \neq 0$ is contained in a finite union of sets of the form $\lambda - Q_+$, $\lambda \in \mathfrak{h}^*$. If M is an $\mathfrak{h}$-semisimple $\mathfrak{g}$-module with finite dimensional weight spaces and weights in a finite union of sets $\lambda - Q_+$ then we can define the **character** of M,

$$\mathrm{ch}(M) = \sum_{\lambda \in \mathfrak{h}^*} \dim M[\lambda] e^\lambda \in \mathcal{R}.$$

For example,

$$\mathrm{ch}(M_\lambda) = \frac{e^\lambda}{\prod_{\alpha \in R_+}(1 - e^{-\alpha})}.$$

We have $\mathrm{ch}(M \otimes N) = \mathrm{ch}(M)\mathrm{ch}(N)$ and

$$\mathrm{ch}(M) = \mathrm{ch}(L) + \mathrm{ch}(N)$$

when $0 \to L \to M \to N \to 0$ is a short exact sequence. Lemma 15.3 implies that we can define such characters $\mathrm{ch}(M)$ for $M \in \mathcal{O}$.

Corollary 15.4. *The action of $Z(\mathfrak{g})$ on every $M \in \mathcal{O}$ factors through a finite dimensional quotient.*

Proof. Since $Z(\mathfrak{g})$ is finitely generated, it suffices to show that every $z \in Z(\mathfrak{g})$ satisfies a polynomial equation $F(z) = 0$ in M. Let $\mu_1, ..., \mu_k$ be weights such that M is generated by $E := M[\mu_1] \oplus ... \oplus M[\mu_k]$. By Lemma 15.3, this space is finite dimensional, and it is preserved by z. Let F be the minimal polynomial of z on E. Then $F(z) = 0$ on E, hence on the whole M (as z is central and E generates M). □

Exercise 15.5. Show that the action of $Z(\mathfrak{g})$ on any Harish-Chandra $(\mathfrak{g}, K)$-module factors through a finite-dimensional quotient. (Mimic the proof of Corollary 15.4).

Exercise 15.6. (i) Show that for any $\mu \in \mathfrak{h}^*$, $\mathrm{Ext}^1_{\mathcal{O}}(M_\mu, M_\mu) = 0$.

(ii) Show that $\mathrm{Ext}^1(M_\mu, M_\mu)$ (Ext in the category of all $\mathfrak{g}$-modules) is nonzero.

Corollary 15.7. *(i) Any $M \in \mathcal{O}$ has a canonical decomposition*

$$M = \oplus_{\chi \in \mathfrak{h}^*/W} M(\chi),$$

where $M(\chi)$ is the generalized eigenspace of $Z(\mathfrak{g})$ in M with eigenvalue χ, and this direct sum is finite. In other words,

$$\mathcal{O} = \oplus_{\chi \in \mathfrak{h}^*/W} \mathcal{O}_\chi,$$

where $\mathcal{O}_\chi$ is the subcategory of $\mathcal{O}$ of modules where every $z \in Z(\mathfrak{g})$ acts with generalized eigenvalue $\chi(z)$.

(ii) Each $M \in \mathcal{O}_\chi$ has a finite filtration with successive quotients having infinitesimal character χ.

Proof. (i) Let $R := Z(\mathfrak{g})/\mathrm{Ann}(M)$ be the quotient of $Z(\mathfrak{g})$ by its annihilator in M. This algebra is finite dimensional, so has the form $R = \prod_{i=1}^m R_i$, where R_i are local with units $\mathbf{e}_i$, corresponding to the generalized eigenvalues $\chi_1, ..., \chi_m \in \mathfrak{h}^*/W$ of $Z(\mathfrak{g})$ on M. So $M = \oplus_{i=1}^m M(\chi_i)$, where $M(\chi_i) := \mathbf{e}_i M$.

(ii) If $M \in \mathcal{O}_\chi$ then the algebra R is local. Let $\mathfrak{m}$ be its unique maximal ideal. Then the required finite filtration on M is

$$M \supset \mathfrak{m}M \supset \mathfrak{m}^2 M...$$

□

Thus the simple objects of $\mathcal{O}_\chi$ are $L_{\mu-\rho}$, where $\chi = \chi_\mu$, i.e., $\mu \in \chi$.

We can partition the W-orbit χ into equivalence classes according to the relation $\mu \sim \nu$ if $\mu - \nu \in Q$. It is clear that this partition defines a decomposition $\mathcal{O}_\chi = \oplus_S \mathcal{O}_\chi(S)$, where S runs over the equivalence classes in χ under the relation $\sim$. Namely, $\mathcal{O}_\chi(S)$ is the subcategory of modules with all weights in $\mu - \rho + Q$, where $\mu \in S$.

Example 15.8. Suppose that $\lambda \in \mathfrak{h}^*$ is such that $w\lambda - \lambda \notin Q$ for any $1 \neq w \in W$. In this case the equivalence relation on $W\lambda$ is trivial, so for any $\mu \in W\lambda$ the category $\mathcal{O}_{\chi_\lambda}(\mu)$ has a unique simple object $M_{\mu-\rho}$. It thus follows from Exercise 15.6 for any $\mu \in W\lambda$, the category $\mathcal{O}_{\chi_\lambda}(\mu)$ is equivalent to the category of finite dimensional vector spaces (as $M_{\mu-\rho}$ has no nontrivial self-extensions), and the category $\mathcal{O}_{\chi_\lambda}$ is semisimple with $|W|$ simple objects.

Lemma 15.9. *Every object of $\mathcal{O}$ has finite length.*

Proof. By Corollary 15.7 we may assume that M has infinitesimal character χ_λ. We may also assume that $P(M) \subset \mu + Q$ for some $\mu \in \mathfrak{h}^*$. Recall that the quadratic Casimir C of $\mathfrak{g}$ acts on M in the same way as in $M_{\lambda-\rho}$, i.e., by the scalar $\lambda^2 - \rho^2$. Suppose that v is a singular vector in a nonzero subquotient M' of M of some weight $\gamma \in \mu + Q$ (it must exist since weights of M' belong to a finite union of $\lambda_i - Q_+$). Then $Cv = (\gamma^2 - \rho^2)v$, so we must have

$$\gamma^2 = \lambda^2.$$

Since the inner product on Q is positive definite, this equation has a finite set S of solutions $\gamma \in \mu + Q$.

For a semisimple $\mathfrak{h}$-module Y set $Y[S] := \oplus_{\gamma \in S} Y[\gamma]$. It follows that $M'[S] \neq 0$. Also by Lemma 15.3 we have $\dim M[S] < \infty$. Thus $\mathrm{length}(M) \leq \dim M'[S] \leq \dim M[S]$ is finite, as claimed. ∎

15.2. **Partial orders of $\mathfrak{h}^*$.** Introduce a partial order on $\mathfrak{h}^*$: we say that $\mu \leq \lambda$ if $\lambda - \mu \in Q_+$ and $\mu < \lambda$ if $\mu \leq \lambda$ but $\mu \neq \lambda$. We write $\lambda \geq \mu$ if $\mu \leq \lambda$ and $\lambda > \mu$ if $\mu > \lambda$.

If $\mu = s_\alpha \lambda$ for some $\alpha \in R_+$ and $\mu < \lambda$ (i.e., $(\lambda, \alpha^\vee) \in \mathbb{Z}_{\geq 1}$ and $\mu = \lambda - (\lambda, \alpha^\vee)\alpha$), then we write $\mu <_\alpha \lambda$. We write $\mu \preceq \lambda$ if there exist sequences $\alpha^1, ..., \alpha^m \in R_+$ and $\mu = \mu_0, \mu_1, ..., \mu_m = \lambda$ such that for all i, $\mu_{i-1} <_{\alpha^i} \mu_i$, and write $\mu \prec \lambda$ if $\mu \preceq \lambda$ but $\mu \neq \lambda$ (i.e., $m \neq 0$). We write $\lambda \succeq \mu$ if $\mu \preceq \lambda$ and $\lambda \succ \mu$ if $\mu \prec \lambda$.

Remark 15.10. It is easy to see that if $\mu \prec \lambda$ then $\mu < \lambda$ and $\mu \in W\lambda$, but the converse is false, in general. For example, consider the root system of type A_3, and let us realize $\mathfrak{h}^*$ as $\mathbb{C}^4/\mathbb{C}_{\mathrm{diagonal}}$. Let $\mu = (0, 3, 1, 2)$, $\lambda = (1, 2, 3, 0)$. Then $\mu \in W\lambda$ and $\mu < \lambda$, since $\lambda - \mu = (1, -1, 2, -2) = \alpha_1 + 2\alpha_3$. However, $\mu \not\prec \lambda$. Indeed, otherwise there would exist $\alpha \in R_+$ such that $\mu \le s_\alpha\lambda < \lambda$, and it is easy to check that there is no such α.

15.3. **Verma's theorem.**

Theorem 15.11. *(D. N. Verma) Let $\lambda, \mu \in \mathfrak{h}^*$ and $\mu \preceq \lambda$. Then $\dim \mathrm{Hom}(M_{\mu-\rho}, M_{\lambda-\rho}) = 1$ and $M_{\mu-\rho}$ can be uniquely realized as a submodule of $M_{\lambda-\rho}$. In particular, $L_{\mu-\rho}$ occurs in the composition series of $M_{\lambda-\rho}$.*

Proof. By Exercise 8.14, $\dim \mathrm{Hom}(M_{\mu-\rho}, M_{\lambda-\rho}) \le 1$ and any nonzero homomorphism $M_{\mu-\rho} \to M_{\lambda-\rho}$ is injective, so it suffices to show that $\dim \mathrm{Hom}(M_{\mu-\rho}, M_{\lambda-\rho}) \ge 1$. By definition of the partial order $\preceq$, it suffices to do so when $\mu <_\alpha \lambda$ for some $\alpha \in R_+$, i.e., when $\mu = s_\alpha\lambda = \lambda - n\alpha$ where $n := (\lambda, \alpha^\vee) \in \mathbb{Z}_+$. For generic λ with $(\lambda, \alpha^\vee) = n \in \mathbb{Z}_+$, this follows from the Shapovalov determinant formula (Exercise 8.15), and the general case follows by taking the limit. $\square$

We will see below that the converse to Verma's theorem also holds: if $L_{\mu-\rho}$ occurs in the composition series of $M_{\lambda-\rho}$ then $\mu \preceq \lambda$. This was proved by J. Bernstein, I. Gelfand and S. Gelfand, see Theorem 20.13 below.

15.4. **The stabilizer in W of a point in $\mathfrak{h}^*/Q$.** Let $x \in \mathfrak{h}^*/Q$ and $W_x \subset W$ be the stabilizer of x.

Proposition 15.12. *W_x is generated by the reflections $s_\alpha \in W_x$. Moreover, the roots α such that $s_\alpha \in W_x$ form a root system $R_x \subset R$, and W_x is the Weyl group of R_x. The corresponding dual root system $R_x^\vee$ is a root subsystem of $R^\vee$, i.e., $R_x^\vee = \mathrm{span}_{\mathbb{Z}}(R_x^\vee) \cap R^\vee$.*

Proof. Let $T := \mathfrak{h}^*/Q$. The ring $\mathbb{C}[T/W] := \mathbb{C}[T]^W$ is freely generated by the orbit sums $m_i = \sum_{\beta \in W\omega_i^\vee} e^\beta$, where $\omega_i^\vee$ are the fundamental coweights. Hence T/W is smooth (in fact, an affine space). It follows by the Chevalley-Shephard-Todd theorem that for each $x \in T$ the stabilizer W_x is generated by a subset of reflections of W. Moreover, if $s_\alpha, s_\beta \in W_x$ then $s_\alpha s_\beta s_\alpha = s_{s_\alpha(\beta)} \in W_x$, which implies that the set R_x of α such that $s_\alpha \in W_x$ is a root system in R, and W_x is its Weyl group. Moreover, picking a preimage $\widetilde{x}$ of x in $\mathfrak{h}^*$, we see that $\alpha \in R_x$ if and only if $(\alpha^\vee, \widetilde{x}) \in \mathbb{Z}$. Thus $R_x^\vee$ is a root subsystem of $R^\vee$. $\square$

Remark 15.13. 1. Note that unlike the case $x \in \mathfrak{h}^*$, for $x \in \mathfrak{h}^*/Q$ the group W_x is not necessarily a **parabolic** subgroup of W, i.e., it is not necessarily conjugate to a subgroup generated by simple reflections. In fact, the Dynkin diagram of R_x or $R_x^\vee$ may not be a subdiagram of the Dynkin diagram of W. Such subgroups are called **quasiparabolic subgroups**.

For example, if R is of type B_2 with simple roots $\alpha_1 = (1,0)$ and $\alpha_2 = (-1,1)$ then for $x = (\frac{1}{2}, 0)$, R_x is the root system of type $A_1 \times A_1$ consisting of $\pm\alpha_1$ and $\pm(\alpha_1 + \alpha_2)$. The same example shows that R_x is not necessarily a root subsystem of R, as $\alpha_1 + (\alpha_1 + \alpha_2) \notin R_x$.

2. If $G^\vee$ is the simply connected complex semisimple Lie group corresponding to $R^\vee$ then T is the maximal torus of $G^\vee$, and it is easy to see that $R_x^\vee$ is the root system of the centralizer $\mathfrak{z}_x$ of x in $\mathfrak{g}^\vee := \mathrm{Lie}(G^\vee)$.

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