16. **Category $\mathcal{O}$ of $\mathfrak{g}$-modules - II**

16.1. **Dominant weights.** Let us say that a weight $\lambda \in \mathfrak{h}^*$ is **dominant** for the partial order $\leq$ (respectively, $\preceq$) if it is maximal with respect to this order in its equivalence class (or, equivalently, in its W-orbit).

Corollary 16.1. *The following conditions on a weight $\lambda \in \mathfrak{h}^*$ are equivalent:*

(i) λ is dominant for $\leq$;
(ii) λ is dominant for $\preceq$;
(iii) For every root $\alpha \in R_+$, $(\lambda, \alpha^\vee) \notin \mathbb{Z}_{<0}$.
(iv) For every $w \in W_{\lambda+Q}$, $w\lambda \preceq \lambda$.
(v) For every $w \in W_{\lambda+Q}$, $w\lambda \leq \lambda$.

Proof. It is clear that (iv) implies (v) implies (i) implies (ii). It is also easy to see that (ii) implies (iii), since if $(\lambda, \alpha^\vee) \in \mathbb{Z}_{<0}$ then $s_\alpha\lambda \sim \lambda$ and $s_\alpha\lambda > \lambda$ so λ is not maximal under $\preceq$ in its equivalence class. It remains to show that (iii) implies (iv). By Proposition 15.12, $W_{\lambda+Q}$ is the Weyl group of some root system $R' \subset R$, and the equivalence class S of λ is simply the orbit $W_{\lambda+Q}\lambda$. By our assumption, for $\alpha \in R'_+$ we have $(\lambda, \alpha^\vee) \in \mathbb{Z} \setminus \mathbb{Z}_{<0} = \mathbb{Z}_{\geq 0}$. Thus, $\lambda = \lambda' + \nu$ where λ' is a dominant integral weight for R' (meaning that $(\lambda, \alpha^\vee) \in \mathbb{Z}_{\geq 0}$ for $\alpha \in R'_+$) and $(\nu, \alpha^\vee) = 0$ for all $\alpha \in R'_+$. Now for any $w \in W_{\lambda+Q}$, fix a reduced decomposition $w = s_{i_m}...s_{i_1}$, where $s_i = s_{\beta_i}$ and β_i are the simple roots of R'. Let $\lambda_k := s_{i_k}...s_{i_1}\lambda$, so $\lambda_0 = \lambda$ and $\lambda_m = w\lambda$. Setting $\lambda'_k := s_{i_k}...s_{i_1}\lambda' = \lambda_k - \nu$, we then have

$$\lambda_{k-1} - \lambda_k = \lambda'_{k-1} - \lambda'_k = (\lambda'_{k-1}, \beta_{i_k}^\vee)\beta_{i_k} = (\lambda', s_{i_1}...s_{i_{k-1}}\beta_{i_k}^\vee)\beta_{i_k}.$$

The coroot $s_{i_1}...s_{i_{k-1}}\beta_{i_k}^\vee$ is positive, so we get that $\lambda_k \preceq \lambda_{k-1}$, which yields (iv). $\square$

Corollary 16.1 shows that every equivalence class of weights contains a unique maximal element with respect to each of the orders $\preceq$ and $\leq$, namely the unique dominant weight in this class. The same is true for minimal elements by changing signs.

16.2. **Projective objects.** Let $\mathcal{C}$ be an abelian category over a field k. Recall that $\mathcal{C}$ is said to be **Noetherian** if any ascending chain of subobjects of any object $X \in \mathcal{C}$ stabilizes. This holds, for instance, when objects of $\mathcal{C}$ have finite length.

Recall also that an object $P \in \mathcal{C}$ is **projective** if the functor $\mathrm{Hom}(P, -)$ is (right) exact, and that $\mathcal{C}$ is said to have **enough projectives** if every object $L \in \mathcal{C}$ is a quotient of a projective object P. Note that if objects of $\mathcal{C}$ have finite length then it is sufficient for this to hold for every simple L, then the property can be proved for all L by induction in length. Indeed, suppose we have a short exact sequence

$$0 \to L_1 \to L \to L_2 \to 0$$

with $L_1, L_2 \neq 0$ and projectives P_1, P_2 with epimorphisms $p_j : P_j \twoheadrightarrow L_j$. Then the map p_2 lifts to $\widetilde{p}_2 : P_2 \to L$, which yields an epimorphism $p_1 + \widetilde{p}_2 : P_1 \oplus P_2 \twoheadrightarrow L$.

Suppose that Hom spaces in $\mathcal{C}$ are finite dimensional. Then by the **Krull-Schmidt theorem**, every object of $\mathcal{C}$ has a unique representation as a finite direct sum of indecomposable ones (up to isomorphism and permutation of summands).

Proposition 16.2. *Let $\mathcal{C}$ be a Noetherian abelian category with enough projectives and finite dimensional Hom spaces over an algebraically closed field k. Then*

(i) Let I be the set labeling the isomorphism classes of indecomposable projectives P_i of $\mathcal{C}$. Then the isomorphism classes of simple objects L_i of $\mathcal{C}$ are labeled by the same set I, and $\dim \mathrm{Hom}(P_i, L_j) = \delta_{ij}$, $i, j \in I$.

(ii) For $M \in \mathcal{C}$ of finite length, the multiplicities $[M : L_i]$ equal $\dim \mathrm{Hom}(P_i, M)$.

Proof. Let $P \in \mathcal{C}$ be an indecomposable projective. Then $\mathrm{End}(P)$ has no idempotents other than $0, 1$, so $\mathrm{End}(P) = k \oplus N$ where N is the nilradical, i.e., it is a local algebra.

Suppose $Q \subset P$ is a maximal proper subobject (it exists by Zorn's lemma since $\mathcal{C}$ is Noetherian). Let $Q' \subset P$ be a subobject not contained in Q. Then $Q + Q' = P$. So we have an epimorphism $Q \oplus Q' \to P$, which, by the projectivity of P, gives a surjection

$$\mathrm{Hom}(P, Q) \oplus \mathrm{Hom}(P, Q') \to \mathrm{End}(P).$$

So we have $1_P = a + a'$, where $a, a' : P \to P$ factor through Q, Q'. Thus a is not an isomorphism (since Q is proper). As $\mathrm{End}(P)$ is local, it follows that a' is an isomorphism, so $Q' = P$.

It follows that P has a unique maximal proper subobject $J(P)$, and $L_P := P/J(P)$ is simple. Moreover, if $L := P/Q$ is simple then $Q = J(P)$, so $L = L_P$. So if I' labels the isomorphism classes of simples in $\mathcal{C}$, then we get a map $\ell : I \to I'$ such that $\ell(P) = L_P$, and we have $\dim \mathrm{Hom}(P_i, L_{i'}) = \delta_{\ell(i),i'}$. Moreover, ℓ is surjective since every simple L is a quotient of some projective P which may be chosen indecomposable (if $P \twoheadrightarrow L$ and $P = \oplus_{i=1}^N P_i$ where P_i are indecomposable then there exists i such that the map $P_i \to L$ is nonzero, hence an epimorphism as L is simple).

It remains to show that ℓ is injective, i.e., if $L_m \cong L_n$ then $P_m \cong P_n$. To this end, note that the epimorphisms $a_0 : P_m \twoheadrightarrow L_n$, $b_0 : P_n \twoheadrightarrow L_m$ lift to morphisms $a : P_m \to P_n$, $b : P_n \to P_m$, such that $ab \in \mathrm{End}(P_n)$ and $ba \in \mathrm{End}(P_m)$ are not nilpotent (as they define isomorphisms on the corresponding simple quotients). Since the algebras $\mathrm{End}(P_n), \mathrm{End}(P_m)$ are loc al, it follows that ab and ba are isomorphisms, as claimed.

This proves (i). Part (ii) now follows from the exactness of the functor $\mathrm{Hom}(P_i, ?)$. $\square$

The object P_i is called the **projective cover** of L_i, and L_i is called the **head** of P_i; by Proposition 16.2, it is the unique simple quotient of P_i.

Remark 16.3. In general, objects of a category satisfying the assumptions of Proposition 16.2 need not have finite length. An example when they can have infinite length is the category of finitely generated $\mathbb{Z}$-graded $\mathbb{C}[x]$-modules, where $\deg(x) = 1$. The simple objects in this category are 1-dimensional modules L_n, $n \in \mathbb{Z}$, which sit in degree n (with x acting by zero). The projective cover of L_n is $P_n = \mathbb{C}[x]_n$, the free rank 1 module sitting in degrees $n, n+1, ...$, which has infinite length.

16.3. Projective objects in $\mathcal{O}$.

Proposition 16.4. *If λ is dominant then $M_{\lambda-\rho}$ is a projective object in $\mathcal{O}$.*[20]

Proof. Our job is to show that the functor $\mathrm{Hom}(M_{\lambda-\rho}, \bullet)$ is exact on $\mathcal{O}$. It suffices to show this on $\mathcal{O}_{\chi_\lambda}(S)$, where S is the equivalence class of λ. To this end, note that all weights of any $X \in \mathcal{O}_{\chi_\lambda}(S)$ are not $> \lambda - \rho$. Thus every $v \in X[\lambda - \rho]$ is singular, so there is a unique homomorphism $M_{\lambda-\rho} \to X$ sending $v_{\lambda-\rho}$ to v. It follows that that $\mathrm{Hom}(M_{\lambda-\rho}, X) \cong X[\lambda - \rho]$, which implies the statement. $\square$

Now let V be a finite dimensional $\mathfrak{g}$-module. Then we have an exact functor $V\otimes : \mathcal{O} \to \mathcal{O}$.

Corollary 16.5. *(i) If $P \in \mathcal{O}$ is projective then so is $V \otimes P$.*

(ii) If $\lambda \in \mathfrak{h}^*$ is dominant then the object $V \otimes M_{\lambda-\rho} \in \mathcal{O}$ is projective.

Proof. (i) For $X \in \mathcal{O}$

$$\mathrm{Hom}_{\mathfrak{g}}(V \otimes P, X) = \mathrm{Hom}_{\mathfrak{g}}(P, V^* \otimes X),$$

[20]Note that this does not mean that M_λ is a projective $U(\mathfrak{g})$-module; in fact, it is not.

which is exact since P is projective.

(ii) follows from (i) and Proposition 16.4. $\square$

Corollary 16.6. *(i) For every $\mu \in \mathfrak{h}^*$, there exists dominant $\lambda \in \mathfrak{h}^*$ and a finite dimensional $\mathfrak{g}$-module V such that $\mathrm{Hom}(V \otimes M_{\lambda-\rho}, L_\mu) \neq 0$. Thus $\mathcal{O}$ has enough projectives.*

(ii) Every projective object P of $\mathcal{O}$ is a free $U(\mathfrak{n}_-)$-module.

Proof. (i) We have

$$\mathrm{Hom}(V \otimes M_{\lambda-\rho}, L_\mu) = \mathrm{Hom}_{\mathfrak{g}}(M_{\lambda-\rho}, V^* \otimes L_\mu).$$

Now take $V = V^* = L_{N\rho}$ for large N and $\lambda = \mu + (N+1)\rho$. It is clear that λ is dominant, and $\mathrm{Hom}_{\mathfrak{g}}(M_{\lambda-\rho}, V^* \otimes L_\mu) = \mathbb{C}$, as claimed.

(ii) This follows by Lemma 12.3 since every indecomposable projective object $P \in \mathcal{O}$ is an $\mathfrak{h}^*$-graded direct summand in $V \otimes M_{\lambda-\rho}$, which is a free graded $U(\mathfrak{n}_-)$-module. $\square$

It follows that every simple object L_λ of $\mathcal{O}$ has a projective cover P_λ, with $\dim \mathrm{Hom}(P_\lambda, L_\mu) = \delta_{\lambda\mu}$.

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