19. **Principal series representations**

19.1. **Residual finiteness of $U(\mathfrak{g})$.**

Proposition 19.1. *The homomorphism $\phi : U(\mathfrak{g}) \to \prod_{\lambda\in P_+} \mathrm{End}(L_\lambda)$ is injective.*

Proof. Let $x \in \mathrm{Ker}\phi$, and G be the simply connected group with Lie algebra $\mathfrak{g}$. Then by the Peter-Weyl theorem, x acts by zero on $\mathcal{O}(G) := \oplus_{\lambda\in P_+} L_\lambda \otimes L_\lambda^*$ (where x acts only on the first component). This means that the right-invariant differential operator on G defined by x is zero, i.e., $x = 0$. □

Exercise 19.2. Give another proof of Proposition 19.1 which does not use the Peter-Weyl theorem. Take $x \in \mathrm{Ker}\phi$.

(i) Show by interpolation that x acts by zero in every Verma module M_λ.

(ii) Show that if $x \in U(\mathfrak{g})$ acts by zero in M_λ for all λ then $x = 0$.

Note that Proposition 19.1 implies that any $z \in U(\mathfrak{g})$ which acts by a scalar in all L_λ belongs to $Z(\mathfrak{g})$. Indeed, in this case for any $x \in U(\mathfrak{g})$, $[x, z]$ acts by zero in L_λ, hence $[x, z] = 0$.

19.2. **Principal series.**

Let $\lambda, \mu \in \mathfrak{h}^*$, $\lambda - \mu \in P$. Define the **principal series** bimodule

$$\mathbf{M}(\lambda, \mu) := \mathrm{Hom}_{\mathrm{fin}}(M_{\lambda-\rho}, M^\vee_{\mu-\rho}) \in HC_{\chi_\mu,\chi_\lambda}(\mathfrak{g}).$$

Then we have

$$\mathbf{M}(\lambda, \mu) = \oplus_{V\in \mathrm{irr}(\mathfrak{g})} V \otimes V^*[\lambda - \mu]. \tag{15}$$

The bimodule $\mathbf{M}(\lambda, \mu)$ represents a certain functor that has a nice independent description.

Proposition 19.3. *Let $X \in HC(\mathfrak{g})$. Then*

$$\mathrm{Hom}_{\mathfrak{g}-\mathrm{bimod}}(X, \mathbf{M}(\lambda, \mu)) \cong \mathrm{Hom}_{(\mathfrak{b}_-,\mathfrak{b}_+)-\mathrm{bimod}}(X \otimes \mathbb{C}_{\lambda-\rho}, \mathbb{C}_{\mu-\rho}).$$

where the $(\mathfrak{b}_-, \mathfrak{b}_+)$-bimodule structure on $\mathbb{C}_{\mu-\rho}$ is defined by the character $(\mu - \rho, 0)$ and on $\mathbb{C}_{\lambda-\rho}$ by the character $(0, \lambda - \rho)$.

Proof. We have

$$\mathrm{Hom}_{\mathfrak{g}-\mathrm{bimod}}(X, \mathbf{M}(\lambda, \mu)) = \mathrm{Hom}_{\mathfrak{g}-\mathrm{bimod}}(X \otimes M_{\lambda-\rho}, M^\vee_{\mu-\rho}),$$

where the right copy of $\mathfrak{g}$ acts trivially on $M^\vee_{\mu-\rho}$ and the left copy of $\mathfrak{g}$ acts trivially on $M_{\lambda-\rho}$. Frobenius reciprocity then yields

$$\mathrm{Hom}_{\mathfrak{g}-\mathrm{bimod}}(X, \mathbf{M}(\lambda, \mu)) = \mathrm{Hom}_{(\mathfrak{b}_+,\mathfrak{g})-\mathrm{bimod}}(X \otimes \mathbb{C}_{\lambda-\rho}, M^\vee_{\mu-\rho}).$$

Since $X \otimes \mathbb{C}_{\lambda-\rho}$ is diagonalizable under the adjoint action of $\mathfrak{h}$, on the right hand side we may replace $M^\vee_{\mu-\rho}$ with its completion $\widehat{M}^\vee_{\mu-\rho}$ (the Cartesian product of all weight spaces). Then applying Frobenius reciprocity again, we get the desired statement. $\square$

Let us give an explicit realization of $\mathbf{M}(\lambda,\mu)$. By (15), $\mathbf{M}(\lambda,\mu)$ is spanned by elements $\Phi_{v,\ell} : M_{\lambda-\rho} \to M^\vee_{\mu-\rho}$, $v \in V, \ell \in V^*[\lambda-\mu]$, where

$$\Phi_{v,\ell} u := (v \otimes 1, \Phi_\ell u),$$

and $\Phi_\ell : M_{\lambda-\rho} \to V^* \otimes M^\vee_{\mu-\rho}$ is the homomorphism for which $\langle \Phi_\ell \rangle = \ell$, for finite dimensional $\mathfrak{g}$-modules V. Moreover these elements easily express in terms of such elements for simple V. Thus for any V and $y \in V \otimes V^*(0)$ we can define the linear map $\Phi_V(y) : M_{\lambda-\rho} \to M^\vee_{\mu-\rho}$ which depends linearly on y with $\Phi_V(v \otimes \ell) = \Phi_{v,\ell}$, and every element of $\mathbf{M}(\lambda,\mu)$ is of this form.

Proposition 19.4. *The right action of $\mathfrak{g}$ on $\mathbf{M}(\lambda,\mu)$ is given by the formula*

$$\Phi_V(v \otimes \ell) \cdot b = \Phi_{\mathfrak{g}\otimes V}([b \otimes v] \bigotimes [(\lambda-\rho) \otimes \ell + \sum_{\alpha \in R_+} f^*_\alpha \otimes f_\alpha \ell]).$$

Proof. Consider the homomorphism

$$\Psi_\ell := \sum_i b^*_i \otimes \Phi_\ell b_i : M_{\lambda-\rho} \to \mathfrak{g}^* \otimes V^* \otimes M^\vee_{\mu-\rho},$$

where $\{b_i\}$ is a basis of $\mathfrak{g}$ and $\{b^*_i\}$ the dual basis of $\mathfrak{g}^*$. We have

$$\langle \Psi_\ell \rangle = \sum b^*_i \otimes \langle \Phi_\ell b_i \rangle \in \mathfrak{g}^* \otimes V^*,$$

where the expectation value map $\langle , \rangle$ is defined in Exercise 8.13. But

$$\langle \Phi_\ell h \rangle = (\lambda-\rho, h)\ell,\ \langle \Phi_\ell e_\alpha \rangle = 0,\ \langle \Phi_\ell f_\alpha \rangle = f_\alpha \ell$$

for $\alpha \in R_+$. Thus we get

$$\langle \Psi_\ell \rangle = (\lambda-\rho) \otimes \ell + \sum_{\alpha \in R_+} f^*_\alpha \otimes f_\alpha \ell,$$

hence

$$\Psi_\ell = \Phi_{(\lambda-\rho)\otimes\ell + \sum_{\alpha\in R_+} f^*_\alpha \otimes f_\alpha \ell}.$$

This implies the statement since

$$(\Phi_V(v \otimes \ell) \cdot b)u = (v \otimes 1, \Phi_\ell bu) = (b \otimes v \otimes 1, \Psi_\ell u),\ u \in M_{\lambda-\rho}.$$

$\square$

This leads to a geometric construction of the principal series. Namely, let G be the simply connected group with Lie algebra $\mathfrak{g}$, $B = B_+$ be the Borel subgroup of G whose Lie algebra is $\mathfrak{b}_+$ and $H = B/[B, B]$ the corresponding torus. Fix $\lambda, \mu \in \mathfrak{h}^*$ with $\lambda - \mu \in P$. Define a real-analytic character

$$\psi_{\lambda,\mu} : H \to \mathbb{C}^\times$$

by

$$\psi_{\lambda,\mu}(x) := \lambda(x)\mu(x^*)^{-1},$$

where x^* is the image of x under the compact antiholomorphic involution $* : H \to H$ (i.e., such that $H^\sigma = H_c$, the compact real form of H). For example, for $G = SL_2$, λ, μ are complex numbers with $\lambda - \mu$ an integer and $x^* = \overline{x}^{-1}$, so

$$\psi_{\lambda,\mu}(x) = x^\lambda \overline{x}^\mu = x^{\lambda-\mu}|x|^{2\mu}.$$

Define $C^\infty_{\lambda,\mu}(G/B)$ to be the space of smooth functions on G satisfying

$$F(gb) = F(g)\psi_{\lambda,\mu}(b).$$

This is naturally an admissible representation of G: we have $G/B = G_c/H_c$, so the multiplicity space of V in $C^\infty_{\lambda,\mu}(G/B)$ is $V^*[\lambda-\mu]$; namely, $C^\infty_{\lambda,\mu}(G/B)^{\text{fin}} = C^\infty_{\lambda-\mu}(G_c/H_c)^{\text{fin}}$, the space of G_c-finite functions on G_c (under left translations) such that

$$F(gx) = F(g)\lambda(x)\mu(x)^{-1}$$

for $x \in H_c$.

Proposition 19.5. *We have an isomorphism*

$$\xi : \mathbf{M}(\lambda, \mu) \to C^\infty_{\lambda-\rho,\mu-\rho}(G/B)^{\text{fin}}$$

as Harish-Chandra bimodules. Namely, $\xi(\Phi_{v,\ell})$ *is the matrix coefficient* $\psi_{v,\ell}(g) := (v, g\ell)$, $g \in G_c$.

Exercise 19.6. Prove Proposition 19.5. **Hint:** Use Proposition 19.4 to show that ξ is a well defined isomorphism of $\mathfrak{g}_{\text{ad}}$-modules, and after applying ξ the right action of $\mathfrak{g}$ looks like

$$(\psi \cdot b)(g) = (\lambda - \rho)(\text{Ad}(g)b)\psi(g) + \sum_{\alpha \in R_+} f^*_\alpha(\text{Ad}(g)b)(R(f_\alpha)\psi)(g),$$

where $R(f_\alpha)$ is the left-invariant vector field equal to f_α at 1. Then show that the right action of $\mathfrak{g}$ on $C^\infty_{\lambda-\rho,\mu-\rho}(G/B)$ is given by the same formula.

19.3. **The functor H_λ.** Define the functor $H_\lambda : \mathcal{O}_\theta \to HC_{\theta,\chi_\lambda}$ given by

$$H_\lambda(X) := \mathrm{Hom}_{\mathrm{fin}}(M_{\lambda-\rho}, X).$$

Note that $H_\lambda(M^\vee_{\mu-\rho}) = \mathbf{M}(\lambda, \mu)$.

Proposition 19.7. *The functor H_λ exact when λ is dominant.*

Proof. If V is a finite dimensional $\mathfrak{g}$-module then

$$\mathrm{Hom}_{\mathfrak{g}}(V, H_\lambda(X)) = \mathrm{Hom}_{\mathfrak{g}}(V \otimes M_{\lambda-\rho}, X),$$

which is exact as $V \otimes M_{\lambda-\rho}$ is projective. □