24. **Applications of projective functors - I**

24.1. **Translation functors.** Let $\theta, \chi \in \mathfrak{h}^*/W$ and V be a finite dimensional irreducible $\mathfrak{g}$-module. Write $F_{\chi,V,\theta}$ for the projective functor $\Pi_\chi \circ F_V \circ \Pi_\theta$, and let us view it as a functor $\mathrm{Rep}(\mathfrak{g})_\theta \to \mathrm{Rep}(\mathfrak{g})_\chi$.

Pick dominant weights $\lambda, \mu \in \mathfrak{h}^*$ such that $\theta = \chi_\lambda, \chi = \chi_\mu$, and $\lambda - \mu \in P$ (this can be done if $F_{\chi,V,\theta} \neq 0$, which we will assume).

Theorem 24.1. *If $W_\lambda = W_\mu$ and V has extremal weight $\mu - \lambda$ then $F_{\chi,V,\theta} : \mathrm{Rep}(\mathfrak{g})_\theta \to \mathrm{Rep}(\mathfrak{g})_\chi$ is an equivalence of categories. A quasi-inverse equivalence is given by the functor $F_{\theta,V^*,\chi}$.*

Proof. It suffices to show that

$$F_{\chi,V,\theta}(M_{\lambda-\rho}) = M_{\mu-\rho},\ F_{\theta,V^*,\chi}(M_{\mu-\rho}) = M_{\lambda-\rho}.$$

Indeed, then

$$F_{\theta,V^*,\chi} \circ F_{\chi,V,\theta}(M_{\lambda-\rho}) = M_{\lambda-\rho},\ F_{\chi,V,\theta} \circ F_{\theta,V^*,\chi}(M_{\mu-\rho}) = M_{\mu-\rho},$$

so

$$F_{\theta,V^*,\chi} \circ F_{\chi,V,\theta} \cong \mathrm{Id}_{\mathrm{Rep}(\mathfrak{g})_\theta},\ F_{\chi,V,\theta} \circ F_{\theta,V^*,\chi} \cong \mathrm{Id}_{\mathrm{Rep}(\mathfrak{g})_\chi},$$

i.e., $F_{\chi,V,\theta}, F_{\theta,V^*,\chi}$ are mutually quasi-inverse equivalences.

We only prove the first statement, the second one being similar. We have

$$F_{\chi,V,\theta}(M_{\lambda-\rho}) = \Pi_\chi(V \otimes M_{\lambda-\rho}).$$

By Corollary 20.5(i), $V \otimes M_{\lambda-\rho}$ has a standard filtration whose composition factors are $M_{\lambda+\beta-\rho}$ where β is a weight of V. The only ones among them that survive the application of Π_χ are those for which $\chi_{\lambda+\beta} = \chi_\mu$, i.e., $\lambda + \beta = w\mu$ for some $w \in W$. So $w\mu \preceq \mu$ (as μ is dominant). Thus, applying Lemma 23.4 with $\phi = \mu, \psi = w\mu$, we get

$$(\lambda - \mu)^2 \le (\lambda - w\mu)^2 = \beta^2.$$

On the other hand, since $\mu - \lambda$ is an extremal weight of V, we have $(\lambda - \mu)^2 \ge \beta^2$. It follows that $(\lambda - \mu)^2 = \beta^2 = (\lambda - w\mu)^2$. Thus by Lemma 23.4 we may choose $w \in W_\lambda$. But since $W_\lambda \subset W_\mu$, it follows that $w\mu = \mu$, so $\beta = \mu - \lambda$. Since the weight multiplicity of an extremal weight is 1, it follows that $F_{\chi,V,\theta}(M_{\lambda-\rho}) = M_{\mu-\rho}$, as claimed. □

Theorem 24.1 shows that for dominant λ the category $\mathrm{Rep}(\mathfrak{g})_{\chi_\lambda}$ depends (up to equivalence) only on the coset $\lambda + P$ and the subgroup $W_\lambda \subset W$. In view of Theorem 24.1, the functors $F_{\chi,V,\theta}$ are called **translation functors** (as they translate between different central characters).

Remark 24.2. Suppose we only have $W_\lambda \subset W_\mu$ instead of $W_\lambda = W_\mu$ (with all the other assumptions being the same). Then the proof of Theorem 24.1 still shows that $F_{\chi,V,\theta}(M_{\lambda-\rho}) = M_{\mu-\rho}$. Thus $[F_{\chi,V,\theta}]\delta_\lambda = \delta_\mu$, and since by Theorem 23.2 $[F_{\chi,V,\theta}]$ is W-invariant, it follows that $[F_{\chi,V,\theta}]\delta_\nu = \delta_\mu$ for all $\nu \in W_\mu\lambda$.

On the other hand, we no longer have $F_{\theta,V^*,\chi}(M_{\mu-\rho}) = M_{\lambda-\rho}$, in general. Namely, the proof of Theorem 24.1 shows that $F_{\theta,V^*,\chi}(M_{\mu-\rho})$ has a filtration whose successive quotients are $M_{\nu-\rho}$, $\nu \in W_\mu\lambda$, each occurring with multiplicity 1 (so the length of this filtration is $|W_\mu/W_\lambda|$). Thus

$$[F_{\theta,V^*,\chi}]\delta_\mu = \sum_{\nu\in W_\mu\lambda} \delta_\nu.$$

It follows that

$$[F_{\chi,V,\theta}][F_{\theta,V^*,\chi}]\delta_\mu = |W_\mu/W_\lambda|\delta_\mu,$$

hence $F_{\chi,V,\theta} \circ F_{\theta,V^*,\chi}(M_{\mu-\rho}) = |W_\mu/W_\lambda|M_{\mu-\rho}$ (as the left hand side is projective). Thus $F_{\chi,V,\theta} \circ F_{\theta,V^*,\chi} \cong |W_\mu/W_\lambda|\mathrm{Id}$.

Remark 24.3. Let $\mathcal{C} \subset \mathrm{Rep}(\mathfrak{g})$ be a full subcategory invariant under all F_V and Π_θ, and $\mathcal{C}_\theta := \Pi_\theta\mathcal{C} = \mathcal{C} \cap \mathrm{Rep}(\mathfrak{g})_\theta$. Then Theorem 24.1 implies that if $W_\lambda = W_\mu$ then the functors $F_{\chi,V,\theta}, F_{\theta,V^*,\chi}$ are mutually quasi-inverse equivalences between $\mathcal{C}_\theta$ and $\mathcal{C}_\chi$. Interesting examples of this include:

1. $\mathcal{C} = \mathcal{O}$. In this case we obtain that for dominant λ the category $\mathcal{O}_{\chi_\lambda}$ up to equivalence depends only on $\lambda + P$ and the stabilizer W_λ. In particular, for regular dominant integral λ all these categories are equivalent.

2. $\mathcal{C}$ is the category of $\mathfrak{g}$-modules which are locally finite and semisimple with respect to a reductive Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$. If $\mathfrak{k}$ is the fixed subalgebra of an involution of $\mathfrak{g}$, this category contains the category of $(\mathfrak{g}_\mathbb{R}, K)$-modules for any connected compact group K such that $\mathrm{Lie}K = \mathfrak{k}$. Namely, it is just the subcategory of modules that integrate to K.

24.2. **Two-sided ideals in U_θ and submodules of Verma modules.** Let $\theta = \chi_\lambda$ for dominant λ. Let Ω_θ denote the lattice of two-sided ideals in U_θ (i.e., the set of two-sided ideals equipped with the operations of sum and intersection). Likewise, let $\Omega(\lambda)$ be the lattice of submodules of $M_{\lambda-\rho}$. We have a map $\nu : \Omega_\theta \to \Omega(\lambda)$ given by $\nu(J) = JM_{\lambda-\rho}$. It is clear that ν preserves inclusion and arbitrary sums.

Theorem 24.4. *(i) $I \subset J$ iff $\nu(I) \subset \nu(J)$. In particular, ν is injective.*

(ii) The image of ν is the set of submodules of $M_{\lambda-\rho}$ which are quotients of direct sums of $P_{\mu-\rho}$ where $\chi_\mu = \chi_\lambda$, $\mu \preceq \lambda$ and $\mu \preceq W_\lambda \mu$.

(iii) If λ is regular (i.e., $W_\lambda = 1$) then ν is an isomorphism of lattices.

Proof. (i) Let F be a projective θ-functor, and $\phi : F \to \mathrm{Id}_\theta$ a morphism of functors $\mathrm{Rep}(\mathfrak{g})^1_\theta \to \mathrm{Rep}(\mathfrak{g})$. Let $M(\phi, F)$ be the image of the map $\phi_{M_{\lambda-\rho}} : F(M_{\lambda-\rho}) \to M_{\lambda-\rho}$ and $J(\phi, F)$ the image of $\phi_{U_\theta} : F(U_\theta) \to U_\theta$. Note that ϕ_{U_θ} is a morphism of $(U(\mathfrak{g}), U_\theta)$-bimodules, so $J(\phi, F)$ is a subbimodule of U_θ, i.e., a 2-sided ideal. Let $a : U_\theta \to M_{\lambda-\rho}$ be the surjection given by $a(u) = uv_{\lambda-\rho}$. Then by functoriality of ϕ

$$a \circ \phi_{U_\theta} = \phi_{M_{\lambda-\rho}} \circ a.$$

Hence

$$\nu(J(\phi, F)) = J(\phi, F)M_{\lambda-\rho} = J(\phi, F)v_{\lambda-\rho} = a(J(\phi, F)) =$$

$$\mathrm{Im}(a \circ \phi_{U_\theta}) = \mathrm{Im}(\phi_{M_{\lambda-\rho}} \circ a) = \mathrm{Im}(\phi_{M_{\lambda-\rho}}) = M(\phi, F).$$

Let us show that any 2-sided ideal J in U_θ is of the form $J(\phi, F)$ for some F, ϕ. Since U_θ is Noetherian, J is generated by some finite dimensional subspace $V \subset J$ which can be chosen $\mathfrak{g}_{\mathrm{ad}}$-invariant. Then by Frobenius reciprocity the $\mathfrak{g}_{\mathrm{ad}}$-morphism $\iota : V \to U_\theta$ can be lifted to a morphism of $(U(\mathfrak{g}), U_\theta)$-bimodules $\widehat{\phi} : V \otimes U_\theta = F_V(U_\theta) \to U_\theta$, i.e., to a functorial morphism $\phi : F_V(\theta) \to \mathrm{Id}_\theta$. It is clear that then $J = J(\phi, F)$.

We are now ready to prove (i), i.e., that $M(\phi, F) \subset M(\phi', F')$ implies $J(\phi, F) \subset J(\phi', F')$. Since $F(M_{\lambda-\rho})$, $F'(M_{\lambda-\rho})$ are projective, the inclusion $M(\phi, F) \hookrightarrow M(\phi', F')$ lifts to a map $\widetilde{\alpha} : F(M_{\lambda-\rho}) \to F'(M_{\lambda-\rho})$, i.e., $\phi'_{M_{\lambda-\rho}} \circ \widetilde{\alpha} = \phi_{M_{\lambda-\rho}}$. But by Theorem 22.4, morphisms of projective θ-functors are the same as morphisms of the images of $M_{\lambda-\rho}$ under these functors. Thus there is $\alpha : F \to F'$ which maps to $\widetilde{\alpha}$ and such that $\phi' \circ \alpha = \phi$. Hence

$$J(\phi, F) = \mathrm{Im}(\phi_{U_\theta}) \subset \mathrm{Im}(\phi'_{U_\theta}) = J(\phi', F'),$$

and (i) follows.

(ii) The proof of (i) implies that the image of ν consists exactly of the submodules $M(\phi, F)$. Such a submodule is the image of $F(M_{\lambda-\rho})$ under a morphism. But F is a projective θ-functor, so by Corollary 22.6(iii), it is of the form $\widetilde{F}(\theta)$, where $\widetilde{F}$ is a projective functor. Also by Theorem 23.6, $\widetilde{F}$ is a direct sum of F_ξ, so $F(M_{\lambda-\rho})$ is a direct sum of $P_{\mu-\rho}$, where (μ, λ) is a proper representation of ξ. Thus $\mu \preceq \lambda$ and $\mu \preceq W_\lambda \mu$. Conversely, if for such μ we have a homomorphism

$\gamma : P_{\mu-\rho} = F_\xi(M_{\lambda-\rho}) \to M_{\lambda-\rho}$ then $\gamma = \phi_{M_{\lambda-\rho}}$ where $\phi : F_\xi(\theta) \to \mathrm{Id}_\theta$. So $\mathrm{Im}(\gamma) = \nu(J(\phi, F_\xi(\theta)))$. Since ν preserves sums, (ii) follows.

(iii) Every submodule of $M_{\lambda-\rho}$ is a quotient of a direct sum of $P_{\mu-\rho}$ with $\chi_\mu = \chi_\lambda, \mu \le \lambda$. Hence by Proposition 16.1 $\mu \preceq \lambda$, as λ is dominant. (This also follows from Theorem 20.13). So if $W_\lambda = 1$ then by (ii) ν is surjective, hence bijective by (i). Since $I \cap J$ is the largest of all ideals contained both in I and in J and similarly for submodules, ν also preserves intersections by (i). Thus ν is an isomorphism of lattices. □

Corollary 24.5. *Let $\theta = \chi_\lambda$ where λ is dominant. If $M_{\lambda-\rho}$ is irreducible then U_θ is a simple algebra. Conversely, if U_θ is simple then $M_{\mu-\rho}$ is irreducible for all μ with $\chi_\mu = \theta$.*

Proof. The direct implication follows from Theorem 24.4. For the reverse implication, suppose for some distinct $\mu_1, \mu_2 \in W\lambda$, we have $M_{\mu_1-\rho} \hookrightarrow M_{\mu_2-\rho}$ and $M_{\mu_1-\rho}$ is simple. Then in view of the Duflo-Joseph theorem we have an inclusion

$$J := \mathrm{Hom}_{\mathrm{fin}}(M_{\mu_2-\rho}, M_{\mu_1-\rho}) \hookrightarrow \mathrm{Hom}_{\mathrm{fin}}(M_{\mu_2-\rho}, M_{\mu_2-\rho}) = U_\theta,$$

and J is a proper 2-sided ideal (as it does not contain 1) which is not zero (as $M_{\mu_1-\rho} \cong M^\vee_{\mu_1-\rho}$ and hence for a finite dimensional $\mathfrak{g}$-module V, $\mathrm{Hom}(M_{\mu_2-\rho}, V \otimes M_{\mu_1-\rho}) \cong V[\mu_2 - \mu_1]$). □

Using the determinant formula for the Shapovalov form, this gives an explicit description of the locus of $\theta \in \mathfrak{h}^*/W$ where U_θ is simple.

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