25. **Applications of projective functors - II**

25.1. **Duflo's theorem on primitive ideals in U_θ.** Recall that a **prime ideal** in a commutative ring R is a proper ideal I such that if $xy \in I$ then $x \in I$ or $y \in I$. This definition is not good for noncommutative rings: for example, the zero ideal in the matrix algebra $\mathrm{Mat}_n(\mathbb{C})$, $n \geq 2$, would not be prime, even though this algebra is simple; so $\mathrm{Mat}_n(\mathbb{C})$ would have no prime ideals at all. However, the definition can be reformulated so that it works well for noncommutative rings.

Definition 25.1. A proper 2-sided ideal I in a (possibly non-commutative) ring R is **prime** if whenever the product XY of two 2-sided ideals $X, Y \subset R$ is contained in I, either X or Y must be contained in I.

Note that for commutative rings this coincides with the usual definition. Indeed, if I is prime in the noncommutative sense and if $xy \in I$ then $(x)(y) \subset I$, so $(x) \subset I$ or $(y) \subset I$, i.e. x or y is in I. Conversely, if I is prime in the commutative sense and X, Y are not contained in I then there exist $x \in X, y \in Y$ not in I, so $xy \notin I$, i.e., XY is not contained in I. But in the noncommutative case the two definitions differ, e.g. 0 is clearly a prime ideal (in the noncommutative sense) in any simple algebra, e.g. in the matrix algebra $\mathrm{Mat}_n(\mathbb{C})$.

A ring R is called **prime** if 0 is a prime ideal in R. For example, if R is an integral domain then it is prime, and the converse holds if R is commutative. On the other hand, there are many noncommutative prime rings which are not domains, e.g. simple rings, such as the matrix algebras $\mathrm{Mat}_n(\mathbb{C}), n \geq 2$. Also it is clear that an ideal $I \subset R$ is prime iff the ring R/I is prime (thus every maximal ideal is prime, so prime ideals always exist). If moreover R/I is a domain, one says that I is **completely prime**.

Another important notion is that of a **primitive ideal**.

Definition 25.2. An ideal $I \subset R$ is **primitive** if it is the annihilator of a simple R-module M.

It is easy to see that every primitive ideal I is prime: if X, Y are 2-sided ideals in R and $XY \subset I$ then $XYM = 0$, so if Y is not contained in I then $YM \neq 0$. Thus $YM = M$ (as M is simple), hence $XM = XYM = 0$, so $X \subset I$. Also for a commutative ring a primitive ideal is the same thing as a maximal ideal. Indeed, if I is maximal then R/I is a field, so a simple R-module, and I is the annihilator of R/I. Conversely, if I is primitive and is the annihilator of a simple module M then $M = R/J$ is a field and $I = J$, so I is maximal.

Exercise 25.3. Show that every maximal ideal in a unital ring is primitive, and give a counterexample to the converse.

We see that in general a prime ideal need not be primitive, e.g. the zero ideal in $\mathbb{C}[x]$. Nevertheless, for U_θ we have the following remarkable theorem due to M. Duflo:

Theorem 25.4. *Every prime ideal $J \subset U_\theta$ is primitive and moreover is the annihilator of a simple highest weight module $L_{\mu-\rho}$, where $\chi_\mu = \theta$.*

Proof. The module $M := M_{\lambda-\rho}/\nu(J)$ has finite length, so let us endow it with a filtration by submodules $F_k = F_k M$ with simple successive quotients $L_1, ..., L_n$ ($L_k = F_k/F_{k-1}$). Let $I_k \subset U_\theta$ be the annihilators of L_k. Since $JM = 0$, we have $J \subset I_k$ for all k. Also $I_k F_k \subset F_{k-1}$, so $I_1...I_n M = 0$, hence $I_1...I_n M_{\lambda-\rho} \subset JM_{\lambda-\rho}$. By Theorem 24.1(i), this implies that $I_1....I_n \subset J$. Since J is prime, this means that there exists m such that $I_m \subset J$. Then $J = I_m$, i.e. J is the annihilator of L_m. But $L_m = L_{\mu-\rho}$ for some μ such that $\chi_\mu = \chi_\lambda = \theta$. □

Note that the choice of μ is not unique, for example, for $J = 0$ and generic θ, any of the $|W|$ possible choices of μ is good. In fact, the proof of Duflo's theorem shows that for every dominant λ such that $\theta = \chi_\lambda$, we can choose $\mu \in W\lambda$ such that $\mu \preceq \lambda$.

25.2. **Classification of simple Harish-Chandra bimodules.** Denote by HC_θ^n the category of Harish-Chandra bimodules over $\mathfrak{g}$ annihilated on the right by the ideal $(\mathrm{Ker}\theta)^n$. These categories form a nested sequence; denote the corresponding nested union by HC_θ. Recall that we have a direct sum decomposition $HC = \oplus_{\theta\in\mathfrak{h}^*/W} HC_\theta$. This implies that every simple Harish-Chandra bimodule belongs to HC_θ^1 for some central character θ.

Recall also that for a finite dimensional $\mathfrak{g}$-module V, in HC_θ^1 we have the object $V \otimes U_\theta$. Moreover, this object is projective: for $Y \in HC_\theta^1$ we have

$$\mathrm{Hom}(V \otimes U_\theta, Y) = \mathrm{Hom}_{\mathfrak{g}-\mathrm{bimod}}(V \otimes U(\mathfrak{g}), Y) = \mathrm{Hom}_{\mathfrak{g}_{\mathrm{ad}}}(V, Y),$$

which is an exact functor since Y is a locally finite (hence semisimple) $\mathfrak{g}_{\mathrm{ad}}$-module. Finally, since Y is a finitely generated bimodule locally finite under $\mathfrak{g}_{\mathrm{ad}}$, there exists a finite dimensional $\mathfrak{g}_{\mathrm{ad}}$-submodule $V \subset Y$ that generates Y as a bimodule. Then the homomorphism

$$\widehat{i} : V \otimes U(\mathfrak{g}) \to Y$$

corresponding to $i : V \hookrightarrow Y$ is surjective and factors through the module $V \otimes U_\theta$. Thus Y is a quotient of $V \otimes U_\theta$. Thus we have

Lemma 25.5. *The abelian category HC^1_θ has enough projectives.*

We also note that this category has finite dimensional Hom spaces. Indeed, If $Y_1, Y_2 \in HC^1_\theta$ then Y_1 is a quotient of $V \otimes U_\theta$ for some V, so $\mathrm{Hom}(Y_1, Y_2) \subset \mathrm{Hom}(V \otimes U_\theta, Y_2) = \mathrm{Hom}_{\mathfrak{g}_{\mathrm{ad}}}(V, Y_2)$, which is finite dimensional. Finally, note that this category is Noetherian: any nested sequence of subobjects of an object stabilizes.

It thus follows from the Krull-Schmidt theorem that in HC^1_θ, every object of HC^1_θ is uniquely a finite direct sum of indecomposables, and from Proposition 16.2 the indecomposable projectives and the simples of HC^1_θ labeled by the same index set. It remains to describe this labeling set.

Theorem 25.6. *The simples (and indecomposable projectives) in HC^1_θ are labeled by the set Ξ, via $\xi \in \Xi \mapsto \mathbf{L}_\xi, \mathbf{P}_\xi$. Namely, if $\xi = (\mu, \lambda)$ is a proper representation then $\mathbf{P}_\xi$ is the unique indecomposable projective in HC^1_θ such that $\mathbf{P}_\xi \otimes_{U(\mathfrak{g})} M_{\lambda-\rho} = P_{\mu-\rho}$.*

Proof. Every indecomposable projective is a direct summand of $V \otimes U_\theta$. But $(V \otimes U_\theta) \otimes_{U(\mathfrak{g})} Y = F_V(\theta)(Y)$. Thus from the classification of projective functors (Theorem 23.6) it follows that the indecomposable summands of $V \otimes U_\theta$ are $\mathbf{P}_\xi$ such that $\mathbf{P}_\xi \otimes = F_\xi(\theta)$. □

Corollary 25.7. *Objects in HC^1_θ, hence in HC_θ and HC, have finite length.*

Proof. Recall that $HC^1_\theta = \oplus_\chi HC^1_{\chi,\theta}$, the decomposition according to left generalized central characters. By Theorem 25.6, each subcategory $HC^1_{\chi,\theta}$ has finitely many simple objects. Thus the statement follows from Proposition 16.2. □

25.3. **Equivalence between category $\mathcal{O}$ and category of Harish-Chandra bimodules.** Let $\theta = \chi_\lambda$ where λ is dominant. Let $\mathcal{O}_{\lambda+P}$ be the full subcategory of $\mathcal{O}$ consisting of modules with weights in $\lambda + P$. Define the functor

$$T_\lambda : HC^1_\theta \to \mathcal{O}_{\lambda+P}$$

given by $T_\lambda(Y) = Y \otimes_{U(\mathfrak{g})} M_{\lambda-\rho}$. Also let $\mathcal{O}(\lambda)$ be the full subcategory of $\mathcal{O}_{\lambda+P}$ of modules M which admit a presentation

$$Q_1 \to Q_0 \to M \to 0,$$

where Q_0, Q_1 are direct sums of $P_{\mu-\rho}$ with $\mu \in \lambda + P$ and $\mu \preceq W_\lambda \mu$.

Note that the functor T_λ is left adjoint to the functor H_λ defined in Subsection 19.3: $H_\lambda(X) = \mathrm{Hom}_{\mathrm{fin}}(M_{\lambda-\rho}, X)$. Indeed,

$$\mathrm{Hom}(T_\lambda(Y), X) = \mathrm{Hom}(Y \otimes_{U(\mathfrak{g})} M_{\lambda-\rho}, X) =$$

$$\mathrm{Hom}(Y, \mathrm{Hom}(M_{\lambda-\rho}, X)) = \mathrm{Hom}(Y, \mathrm{Hom}_{\mathrm{fin}}(M_{\lambda-\rho}, X)) = \mathrm{Hom}(Y, H_\lambda(X)).$$

Theorem 25.8. *(J. Bernstein-S, Gelfand) (i) If λ is a regular weight then the functor T_λ is an equivalence of categories, with quasi-inverse H_λ.*

(ii) In general, T_λ is fully faithful and defines an equivalence

$$HC^1_\theta \cong \mathcal{O}(\lambda),$$

with quasi-inverse H_λ.

Remark 25.9. Note that if λ is not regular then the subcategory $\mathcal{O}(\lambda) \subset \mathcal{O}$ need not be closed under taking subquotients (even though it is abelian by Theorem 25.8). Also the functor T_λ (and thus the inclusion $\mathcal{O}(\lambda) \hookrightarrow \mathcal{O}$) need not be (left) exact. So if $f : X \to Y$ is a morphism in $\mathcal{O}(\lambda)$ then its kernels in $\mathcal{O}(\lambda)$ and in $\mathcal{O}$ may differ, and in particular the latter may not belong to $\mathcal{O}(\lambda)$. See Example 26.2.

Proof. (i) is a special case of (ii), so let us prove (ii). To this end, we'll use the following general fact.

Proposition 25.10. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough projectives and $T : \mathcal{A} \to \mathcal{B}$ a right exact functor which maps projectives to projectives. Suppose that T is fully faithful on projectives, i.e., for any projectives $P_0, P_1 \in \mathcal{A}$, the natural map* $\mathrm{Hom}(P_1, P_0) \to \mathrm{Hom}(T(P_1), T(P_0))$ *is an isomorphism. Then T is fully faithful, and defines an equivalence of $\mathcal{A}$ onto the subcategory of objects $Y \in \mathcal{B}$ which admit a presentation*

$$T(P_1) \to T(P_0) \to Y \to 0$$

for some projectives $P_0, P_1 \in \mathcal{A}$.

Proof. We first show that T is faithful. Let $X, X' \in \mathcal{A}$ and $a : X \to X'$. Pick presentations

$$P_1 \to P_0 \to X \to 0, \;\; P_1' \to P_0' \to X' \to 0.$$

We have maps $p_0 : P_0 \to X$, $p_0' : P_0' \to X'$, $p_1 : P_1 \to P_0$, $p_1' : P_1' \to P_0'$. There exist morphisms $a_0 : P_0 \to P_0'$, $a_1 : P_1 \to P_1'$ such that (a_1, a_0, a) is a morphism of presentations.

Suppose $T(a) = 0$. Then $T(p_0')T(a_0) = 0$. Thus $Y := \mathrm{Im}T(a_0) \subset \mathrm{Ker}T(p_0') = \mathrm{Im}T(p_1')$. Thus the map $T(a_0) : T(P_0) \to Y$ lifts to $b : T(P_0) \to T(P_1')$ such that $T(a_0) = T(p_1')b$. Since T is full on projectives, we have $b = T(c)$ for some $c : P_0 \to P_1'$, so $T(a_0) = T(p_1')T(c) = T(p_1'c)$. Since T is faithful on projectives, this implies that $a_0 = p_1'c$. Thus $\mathrm{Im}a_0 \subset \mathrm{Im}p_1' = \mathrm{Ker}p_0'$. It follows that $p_0'a_0 = 0$, hence $ap_0 = 0$. But p_0 is an epimorphism, hence $a = 0$, as claimed.

Now let us show that T is full. Let $X, X' \in \mathcal{A}$ and $b : T(X) \to T(X')$. The functor T maps the above presentations of X, X' into presentations of $T(X), T(X')$ (as it is right exact and maps projectives to projectives):

$$T(P_1) \to T(P_0) \to T(X) \to 0,\ T(P_1') \to T(P_0') \to T(X') \to 0,$$

and we can find $b_0 : T(P_0) \to T(P_0'), b_1 : T(P_1) \to T(P_1')$ such that (b_1, b_0, b) is a morphism of presentations. Since T is fully faithful on projectives, there exist a_0, a_1 such that $T(a_0) = b_0, T(a_1) = b_1$ and $a_0 p_1 = p_1' a_0$. Thus a_0 maps $\mathrm{Im} p_1 = \mathrm{Ker} p_0$ into $\mathrm{Im} p_1' = \mathrm{Ker} p_0'$. This implies that a_0 descends to $a : X \to X'$, and $T(a)T(p_0) = T(p_0')b_0$. Hence $(T(a) - b)T(p_0) = 0$, so since $T(p_0)$ is an epimorphism we get $T(a) = b$, as claimed.

If $Y \in \mathrm{Im}(T)$ then $Y = T(X)$ where X has presentation

$$P_1 \to P_0 \to X \to 0.$$

Thus Y has presentation

$$T(P_1) \to T(P_0) \to Y \to 0.$$

Conversely, if Y has such a presentation as a cokernel of a morphism $f : T(P_1) \to T(P_0)$ then $f = T(g)$ where $g : P_1 \to P_0$, and $Y = T(\mathrm{Coker}(g))$, which proves the last claim of the proposition. □

Now we are ready to prove Theorem 25.8. By Lemma 25.5, HC_θ^1 has enough projectives. Also the functor T_λ is right exact, as it is given by tensor product. Further, if P is projective then $\mathrm{Hom}(T_\lambda(P), Y) = \mathrm{Hom}(P, H_\lambda(Y))$ is exact in Y since H_λ is exact by Proposition 19.7 and P is projective. Thus $T_\lambda(P)$ is projective. Finally, the fact that T_λ is fully faithful on projectives was one of the main results about projective functors (Theorem 22.4). So Proposition 25.10 applies to $\mathcal{A} = HC_\theta^1$, $\mathcal{B} = \mathcal{O}_{\lambda+P}$, $T = T_\lambda$. Moreover, the image of T_λ is precisely the category $\mathcal{O}(\lambda)$ by the classification of projective functors (Theorem 23.6).

For an equivalence of categories, a right adjoint is a quasi-inverse. Thus H_λ is quasi-inverse of T_λ, as claimed. The theorem is proved. □

Corollary 25.11. *Every Harish-Chandra bimodule M with right central character θ is realizable as $\mathbb{V}^{\mathrm{fin}}$ where $\mathbb{V}$ is a (not necessarily unitary) admissible representation of the complex simply connected group G corresponding to $\mathfrak{g}$ on a Hilbert space.*

Proof. Let us prove the statement if $\theta = \chi_\lambda$ where λ is a regular dominant weight (the general proof is similar).

We have seen in Subsection 19.3 that $H_\lambda(M_{\mu-\rho}^\vee)$ is the principal series module $\mathbf{M}(\lambda, \mu) = \mathrm{Hom}_{\mathrm{fin}}(M_{\lambda-\rho}, M_{\mu-\rho}^\vee)$. Thus by Theorem 25.8

$\mathbf{M}(\lambda, \mu)$ is injective in $HC_\theta^!$ if μ is dominant (since $M_{\mu-\rho}$ is projective, hence $M_{\mu-\rho}^\vee$ is injective). Moreover, since every indecomposable projective in $\mathcal{O}_{\lambda+P}$ is a direct summand of $V \otimes M_{\mu-\rho}$ for some dominant μ and finite dimensional $\mathfrak{g}$-module V, it follows that every indecomposable injective is a direct summand in $V \otimes M_{\mu-\rho}^\vee$ for some V and dominant μ. Hence by Theorem 25.8, every indecomposable injective in HC_θ^1 is a direct summand in $V \otimes \mathbf{M}(\lambda, \mu)$ for some V and dominant μ. Thus any $Y \in HC_1^\theta$ is contained in a direct sum of objects $V \otimes \mathbf{M}(\lambda, \mu)$ for finite dimensional V and dominant μ. Since principal series modules $\mathbf{M}(\lambda, \mu)$ are realizable in a Hilbert space by Proposition 19.5, we are done by Corollary 6.13. $\square$

Exercise 25.12. (i) Generalize the proof of Corollary 25.11 to non-regular dominant weights λ.

(ii) Generalize Corollary 25.11 to any Harish-Chandra bimodule with *generalized* central character θ, and then to any Harish-Chandra bimodule.

Hint. Recall that $C^\infty_{\lambda,\mu}(G/B)$ is the space of smooth functions F on G which satisfy the differential equations

$$(R_b - \lambda(b))F = (R_{\bar{b}} - \mu(\bar{b}))F = 0$$

for $b \in \mathfrak{b}$ and $\bar{b} \in \overline{\mathfrak{b}}$ (here R_b is the vector field corresponding to the right translation by b). Now for $N \geq 1$ consider the space $C^\infty_{\lambda,\mu,N}(G/B)$ of smooth functions F on G satisfying the differential equations

$$(R_b - \lambda(b))^N F = (R_{\bar{b}} - \mu(\bar{b}))^N F = 0.$$

(Note that $C^\infty_{\lambda,\mu,1}(G/B) = C^\infty_{\lambda,\mu}(G/B)$.) Show that $C^\infty_{\lambda,\mu,N}(G/B)$ are admissible representations of G on Fréchet spaces. Then mimic the proof of Corollary 25.11 using these instead of $C^\infty_{\lambda,\mu}(G/B)$.

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