7 Derivative of Matrix Determinant and Inverse

7.1 Two Derivations

This section of notes follows this Julia notebook. This notebook is a little bit short, but is an important and useful calculation.

Theorem 39
Given A is a square matrix, we have

$$\nabla(\det A) = \text{cofactor}(A) = (\det A)A^{-T} := \text{adj}(A^T) = \text{adj}(A)^T$$

where adj is the "adjugate". (You may not have heard of the matrix adjugate, but this formula tells us that it is simply $\text{adj}(A) = \det(A)A^{-1}$, or $\text{cofactor}(A) = \text{adj}(A^T)$.) Furthermore,

$$d(\det A) = \text{tr}(\det(A)A^{-1}dA) = \text{tr}(\text{adj}(A)dA) = \text{tr}(\text{cofactor}(A)^T dA).$$

You may remember that each entry (i,j) of the cofactor matrix is $(-1)^{i+j}$ times the determinant obtained by deleting row i and column j from A. Here are some 2×2 calculuations to obtain some intuition about these functions:

$$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \tag{4}$$

$$\implies \text{cofactor}(M) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \tag{5}$$

$$\text{adj}(M) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{6}$$

$$(M)^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{7}$$

Numerically, as is done in the notebook, you can construct a random $n \times n$ matrix A (say, 9×9), consider e.g. $dA = .00001A$, and see numerically that

$$\det(A + dA) - \det(A) \approx \text{tr}(\text{adj}(A)dA),$$

which numerically supports our claim for the theorem.

We now prove the theorem in two ways. Firstly, there is a direct proof where you just differentiate the scalar with respect to every input using the cofactor expansion of the determinant based on the i-th row. Recall that

$$\det(A) = A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in}.$$

Thus,

$$\frac{\partial \det A}{\partial A_{ij}} = C_{ij} \implies \nabla(\det A) = C,$$

the cofactor matrix. (In computing these partial derivatives, it's important to remember that the cofactor C_{ij} contains no elements of A from row i or column j. So, for example, A_{i1} only appears explicitly in the first term, and not hidden in any of the C terms in this expansion.)

There is also a fancier proof of the theorem using linearization near the identity. Firstly, note that it is easy to see from the properties of determinants that

$$\det(I + dA) - 1 = \operatorname{tr}(dA),$$

and thus

$$\begin{aligned}\det(A + A(A^{-1}dA)) - \det(A) &= \det(A)(\det(I + A^{-1}dA) - 1) \\ &= \det(A)\operatorname{tr}(A^{-1}dA) = \operatorname{tr}(\det(A)A^{-1}dA) \\ &= \operatorname{tr}(\operatorname{adj}(A)dA).\end{aligned}$$

This also implies the theorem.

7.2 Applications

7.2.1 Characteristic Polynomial

We now use this as an application to find the derivative of a characteristic polynomial evaluated at x. Let $p(x) = \det(xI - A)$, a scalar function of x. Recall that through factorization, $p(x)$ may be written in terms of eigenvalues λ_i. So we may ask: what is the derivative of $p(x)$, the characteristic polynomial at x? Using freshman calculus, we could simply compute

$$\frac{d}{dx}\prod_i (x - \lambda_i) = \sum_i \prod_{j \neq i}(x - \lambda_j) = \prod (x - \lambda_i)\{\sum_i (x - \lambda_i)^{-1}\},$$

as long as $x \neq \lambda_i$.

This is a perfectly good simply proof, but with our new technology we have a new proof:

$$\begin{aligned}d(\det(xI - A)) &= \det(xI - A)\operatorname{tr}((xI - A)^{-1}d(xI - A)) \\ &= \det(xI - A)\operatorname{tr}(xI - A)^{-1}dx.\end{aligned}$$

Note that here we used that $d(xI - A) = dx\, I$ when A is constant and $\operatorname{tr}(Adx) = \operatorname{tr}(A)dx$ since dx is a scalar.

We may again check this computationally as we do in the notebook.

7.2.2 The Logarithmic Derivative

We can similarly compute using the chain rule that

$$d(\log(\det(A))) = \frac{d(\det A)}{\det A} = \det(A^{-1})d(\det(A)) = \operatorname{tr}(A^{-1}dA).$$

The logarithmic derivative shows up a lot in applied mathematics. Note that here we use that $\frac{1}{\det A} = \det(A^{-1})$ as $1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$.

For instance, recall Newton's method to find roots $f(x) = 0$ of single-variable real-valued functions $f(x)$ by taking a sequence of steps $x \to x + \delta x$. The key formula in Newton's method is $\delta x = f'(x)^{-1}f(x)$, but this is the same as $\frac{1}{(\log f(x))'}$. So, derivatives of log determinants show up in finding *roots of determinants*, i.e. for $f(x) = \det M(x)$. When $M(x) = A - xI$, roots of the determinant are eigenvalues of A. For more general functions $M(x)$, solving $\det M(x) = 0$ is therefore called a *nonlinear eigenproblem*.

7.3 Jacobian of the Inverse

Lastly, we compute the derivative (as both a linear operator and an explicit Jacobian matrix) of the inverse of a matrix. There is a neat trick to obtain this derivative, simply from the property $A^{-1}A = I$ of the inverse. By the product rule, this implies that

$$\begin{aligned} d(A^{-1}A) = d(I) = 0 &= d(A^{-1})A + A^{-1}dA \\ \implies & \boxed{d(A^{-1}) = (A^{-1})'[dA] = -A^{-1}\, dA\, A^{-1}}. \end{aligned}$$

Here, the second line defines a perfectly good linear operator for the derivative $(A^{-1})'$, but if we want we can rewrite this as an explicit Jacobian matrix by using Kronecker products acting on the "vectorized" matrices as we did in Sec. 3:

$$\mathrm{vec}\left(d(A^{-1})\right) = \mathrm{vec}\left(-A^{-1}(dA)A^{-1}\right) = \underbrace{-(A^{-T} \otimes A^{-1})}_{\text{Jacobian}} \mathrm{vec}(dA)\,,$$

where A^{-T} denotes $(A^{-1})^T = (A^T)^{-1}$. One can check this formula numerically, as is done in the notebook.

In practice, however, you will probably find that the operator expression $-A^{-1}\, dA\, A^{-1}$ is more useful than explicit Jacobian matrix for taking derivatives involving matrix inverses. For example, if you have a matrix-valued function $A(t)$ of a scalar parameter $t \in \mathbb{R}$, you immediately obtain $\frac{d(A^{-1})}{dt} = -A^{-1}\frac{dA}{dt}A^{-1}$. A more sophisticated application is discussed in Sec. 6.3.