2.001 - MECHANICS AND MATERIALS I
Lecture #15
11/1/2006
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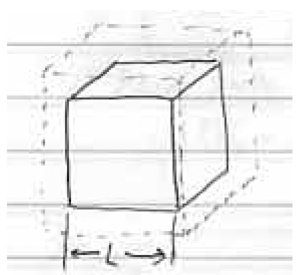
Recall equations of isotropic linear elasticity:

$$\epsilon_{xx} = \frac{1}{E}\Big[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\Big]$$
$$\epsilon_{yy} = \frac{1}{E}\Big[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\Big]$$
$$\epsilon_{xx} = \frac{1}{E}\Big[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\Big]$$
$$\epsilon_{xy} = \frac{1}{2G}\sigma_{xy}$$
$$\epsilon_{xz} = \frac{1}{2G}\sigma_{xz}$$
$$\epsilon_{yz} = \frac{1}{2G}\sigma_{yz}$$

Thermoelastic Behavior:



$$\Delta T > 0$$
$$L \rightarrow L + \alpha \Delta T L$$

α is the coefficient of linear thermal expansion "CTE".

Thermal Strain (For an unconstrained block)

$$\epsilon^T_{xx} = \alpha \Delta T$$
$$\epsilon^T_{yy} = \alpha \Delta T$$
$$\epsilon^T_{zz} = \alpha \Delta T$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{xz} = 0$$

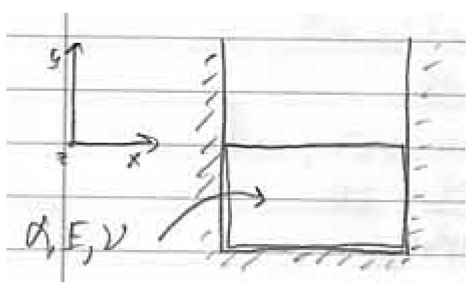
Note: If the block were constrained, there could be a "thermal stress."
Total Strain:

$$\epsilon_{xx} = \epsilon_{xx}^{\text{Elastic}} + \epsilon_{xx}^{\text{Thermal}}$$
$$\epsilon_{yy} = \epsilon_{yy}^{\text{E}} + \epsilon_{yy}^{\text{T}}$$
$$\epsilon_{zz} = \epsilon_{zz}^{\text{E}} + \epsilon_{zz}^{\text{T}}$$

Equations of Linear, Isotropic, Thermoelasticity

$$\epsilon_{xx} = \frac{1}{E}\Big[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\Big] + \alpha \Delta T$$
$$\epsilon_{yy} = \frac{1}{E}\Big[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\Big] \alpha \Delta T$$
$$\epsilon_{xx} = \frac{1}{E}\Big[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\Big] \alpha \Delta T$$
$$\epsilon_{xy} = \frac{1}{2G}\sigma_{xy}$$
$$\epsilon_{xz} = \frac{1}{2G}\sigma_{xz}$$
$$\epsilon_{yz} = \frac{1}{2G}\sigma_{yz}$$

EXAMPLE: Block in a frictionless channel



Subject block to an increased T, $\Delta T > 0$
Find ϵ^T, σ

$\epsilon_{xx} = 0 \quad \sigma_{xx} = ?$
$\epsilon_{yy} = ? \quad \sigma_{yy} = 0$
$\epsilon_{zz} = ? \quad \sigma_{zz} = 0$

Use equations of linear, isotropic, thermoelasticity

$$\epsilon_{xx} = \frac{1}{E}\Big[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\Big] + \alpha \Delta T$$
$$\sigma_{yy} \to 0$$
$$\sigma_{zz} \to 0$$

$$0 = \frac{\sigma_{xx}}{E} + \alpha \Delta T \Rightarrow \sigma_{xx} = -\alpha \Delta T E$$

$$\epsilon_{yy} = \frac{1}{E}\Big[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\Big] + \alpha \Delta T$$

$$\epsilon_{yy} = \Big[\frac{-\nu}{E}\sigma_{xx} + \alpha \Delta T \Rightarrow \epsilon_{yy} = \frac{-\nu}{E}(-\alpha \Delta T E) + \alpha \Delta T$$

$$\epsilon_{zz} = \frac{1}{E}\Big[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\Big] + \alpha \Delta T$$

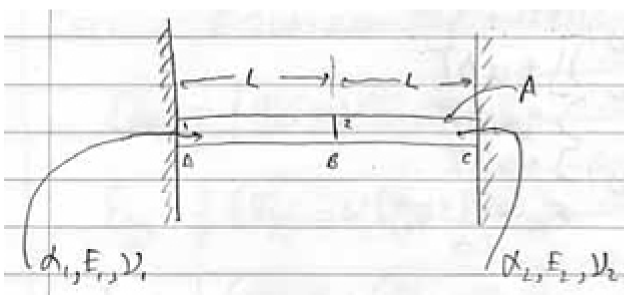
$$\epsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} + \alpha \Delta T \Rightarrow \epsilon_{zz} = \frac{-\nu}{E}(-\alpha \Delta T E) + \alpha \Delta T$$

So:

$$\epsilon_{yy} = \epsilon_{zz} = (\nu + 1)\alpha \Delta T$$

Superposition of the thermal strain and poisson effect from the "thermal stress" (σ_{xx}).

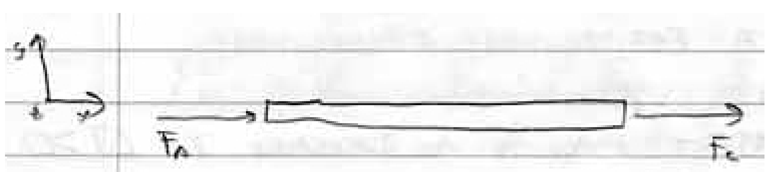
EXAMPLE:



Subject bar to increased T, $\Delta T > 0$

Q: What is the displacement ($\vec{u}$) of B.

FBD:



Equilibrium:

$$\sum F_x = 0$$

$$F_A + F_C = 0$$

$$F_A = -F_C$$

2 Unknowns, 1 Equation

Force-Deformation Relationships (Constitutive Relationships)

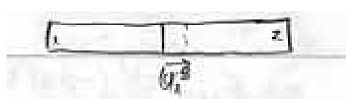
$$\epsilon_{xx_1} = \frac{1}{E_1}\left(\sigma_{xx_1} - \nu_1(\sigma_{yy_1} + \sigma_{zz_1})\right) + \alpha_1 \Delta T$$

$$\epsilon_{xx_1} = \frac{1}{E_1}\sigma_{xx_1} + \alpha_1 \Delta T$$

$$\epsilon_{xx_2} = \frac{1}{E_2}\left(\sigma_{xx_2} - \nu_2(\sigma_{yy_2} + \sigma_{zz_2})\right) + \alpha_2 \Delta T$$

$$\epsilon_{xx_2} = \frac{1}{E_2}\sigma_{xx_2} + \alpha_2 \Delta T$$

Compatibility



$$\delta_1 = u_x^B$$

$$\delta_2 = -u_x^B$$

So:

$$\delta_1 = -\delta_2$$

Recall, for uniaxially loaded bar:

$$\epsilon_{xx} = \frac{\delta}{L}$$

So:

$$\delta = \epsilon_{xx} L$$

Thus:

$$\delta_1 = \epsilon_{xx_1} L$$

$$\delta_2 = \epsilon_{xx_2} L$$

So:

$$\epsilon_{xx_1} L = -\epsilon_{xx_2} L$$
$$\epsilon_{xx_1} = -\epsilon_{xx_2}$$

Also, for uniaxially loaded bar:

$$\sigma_{xx} = \frac{P}{A}$$

So:

$$F_A = -\sigma_{xx_1} A$$
$$F_B = \sigma_{xx_1} A$$

Rewritten Equilibrium:

$$-\sigma_{xx_1} A = -\sigma_{xx_2} A$$
$$\sigma_{xx_1} = \sigma_{xx_2} \Leftarrow \text{ Equilibrium}$$

Substituting Compatibility into Force-Deformation:

$$-\epsilon_{xx_1} = \frac{1}{E_1}\sigma_{xx_1} + \alpha_1 \Delta T$$
$$-\epsilon_{xx_2} = \frac{1}{E_2}\sigma_{xx_2} + \alpha_2 \Delta T$$

Solve for σ_{xx_1}:

$$\sigma_{xx_1} = -E_1(\epsilon_{xx_1} + \alpha_1 \Delta T)$$

Substitute into equilibrium:

$$\sigma_{xx_2} = -E_1(\epsilon_{xx_2} + \alpha_1 \Delta T)$$

Back substitute into force-deformation:

$$\epsilon_{xx_2} = \frac{1}{E_2}(-E_1(\epsilon_{xx_2} + \alpha_1 \Delta T)) + \alpha_2 \Delta T$$
$$\epsilon_{xx_2} = \frac{-E_1}{E_2}\epsilon_{xx_2} - \frac{E_1}{E_2}\alpha_1 \Delta T + \alpha_2 \Delta T$$

$$\left(1+\frac{E_1}{E_2}\right)\epsilon_{xx_2} = -\frac{E_1}{E_2}\alpha_1\Delta T + \alpha_2\Delta T$$

$$\frac{E_1+E_2}{E_2}\epsilon_{xx_2} = \frac{-E_1}{E_2}\alpha_1\Delta T + \alpha_2\Delta T = \left(\frac{-E_1}{E_2}\alpha_1 + \alpha_2\right)\Delta T$$

$$\epsilon_{xx_2} = \frac{E_2}{E_1+E_2}\left(\frac{E_2\alpha_2 - E_1\alpha_1}{-E_2}\right)\Delta T$$

$$\epsilon_{xx_2} = \frac{E_2\alpha_2 - E_1\alpha_2}{E_1+E_2}\Delta T$$

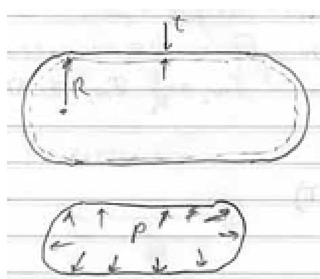
Recall $\delta - \epsilon$ Relationship:

$$\delta_2 = \epsilon_{xx_2} L = \frac{E_2\alpha_2 - E_1\alpha_1}{E_1+E_2}\Delta T L$$

So:

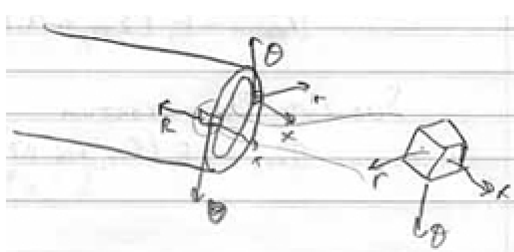
$$u_x^B = \frac{E_1\alpha_1 - E_2\alpha_2}{E_1+E_2}\Delta T L$$

Thin Walled Pressure Vessels



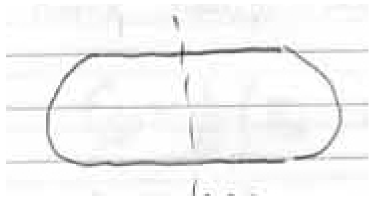
$$t << R \Leftarrow \text{ Thin Walled Vessel}$$

Soda cans, pipes, balloons, etc.

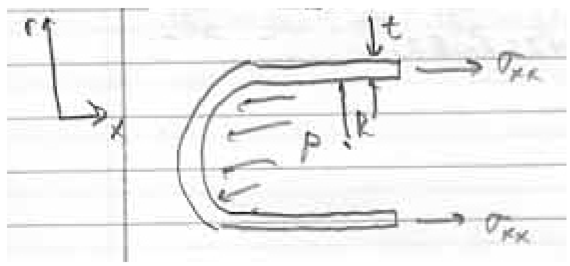


Approximations for thin-walled:
    $r \approx R$ within the walls

$\sigma \approx$ uniform through the wall thickness
$\epsilon \approx$ uniform through the wall thickness



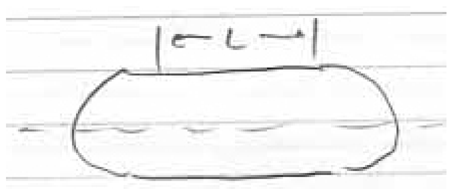
FBD of End Caps



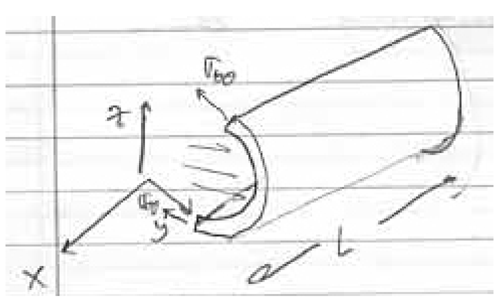
$$\sum F_x = 0$$

$$\sigma_{xx}(2\pi R t) - P(\pi R^2) = 0$$

$$\sigma_{xx} = \frac{PR}{2t} \text{ Axial Stress}$$

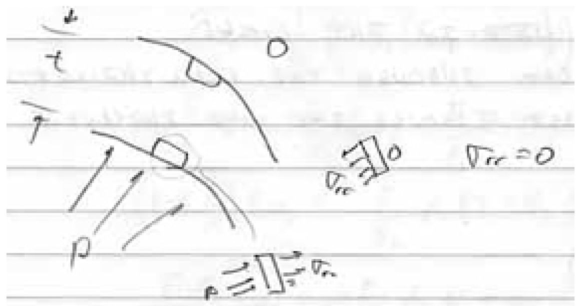


Ignore ends:
FBD:



$$\sum F_y = 0$$
$$-\sigma_{\theta\theta} Lt(2) + P(2R)L = 0$$
$$\sigma_{\theta\theta} = \frac{PR}{t} \text{ Hoop Stress}$$

Stress in radial direction:



$$\sigma_{rr} = -p$$

So anywhere through thickness:

$$|(\sigma_{rr})_m ax| = p$$

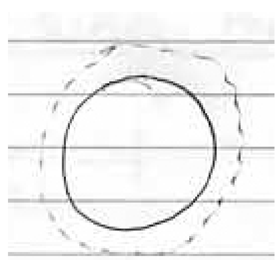
Note: $\sigma_{rr} << \sigma_{xx}, \sigma_{\theta\theta}$.

$p << \frac{pR}{t}$ since $R >> t$, $\frac{R}{t} >> 1$.

So approximate $\sigma_{rr} \approx 0 \Rightarrow$. This is in plane-stress.

What about strain in a thin walled cylinder?

$$\epsilon_{\theta\theta} \neq \frac{\partial u_\theta}{\partial \theta}$$



$$\epsilon_{\theta\theta} = \frac{\Delta \text{ circumference}}{\text{circumference}} = \frac{2\pi \Delta R}{2\pi R}$$

$$\epsilon_{\theta\theta} = \frac{\Delta R}{R}$$

$$\epsilon_{xx} = \frac{\Delta L}{L}$$

$$\epsilon_{rr} = \frac{\Delta t}{t}$$

$$[\sigma\ ] = \begin{vmatrix} 0 & 0 & 0 \\ 0 & \frac{Rp}{t} & 0 \\ 0 & 0 & \frac{pR}{2t} \end{vmatrix}.$$

Apply constitutive relationships.

$$\epsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu(\sigma_{xx} + \sigma_{rr})) + \alpha\Delta T$$

$$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu(\sigma_{\theta\theta} + \sigma_{rr})) + \alpha\Delta T$$

$$\sigma_{rr} \to 0$$