2.003 Engineering Dynamics
Problem Set 6
See appendix at the end of the problem set on how to compute T and V.

Problem 1: A slender uniform rod of mass m_2 is attached to a cart of mass m_1 at a frictionless pivot located at point „A". The cart is connected to a fixed wall by a spring and a damper. The cart rolls without friction in the horizontal direction. The position of the cart in the inertial frame O_{xyz} is given by $x\hat{i}$.

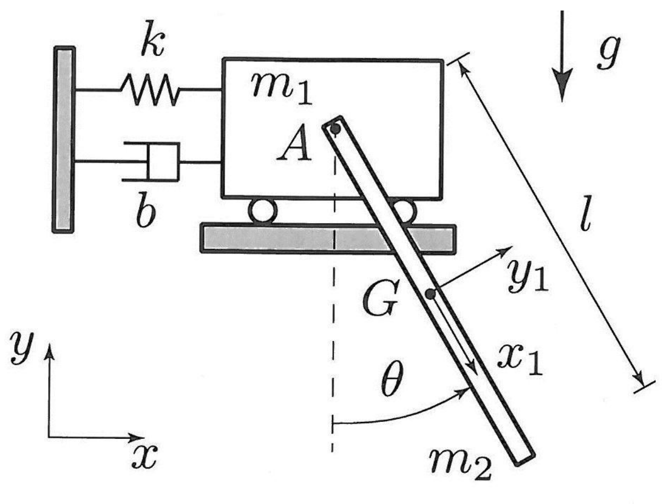


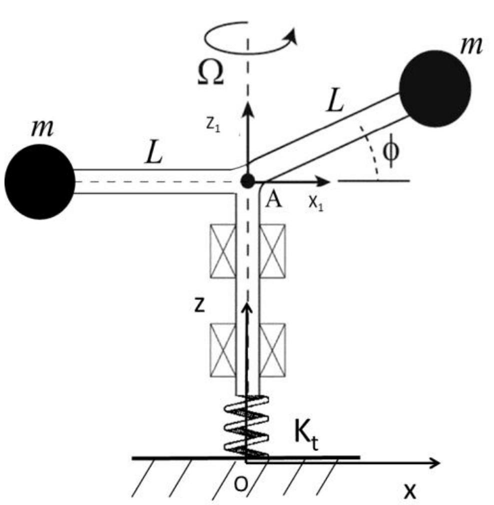
Figure 1. Cart with a slender rod

a). Assuming that the motion of the cart and slender rod is the result of initial conditions only, find expressions for T and V, the kinetic and potential energy of the system in terms of $x, \dot{x}, \theta, \text{and} \dot{\theta}$.

Concept question: The kinetic energy of the rod may be expressed by the equation $T_{rod} = \frac{1}{2} I_{zz/A} \dot{\theta}^2$: a). True, b). False.

Problem 2:

Two identical masses are attached to the end of massless rigid arms as shown in the figure. The vertical portion of the rod is held in place by bearings that prevent vertical motion, but allow the shaft to rotate without friction. A torsion spring with spring constant K_t resists rotation of the vertical shaft. The shaft rotates with a time varying angular velocity Ω with respect to the O_{xyz} inertial frame. The arms are of length L. The frame A_{x1y1z1} rotates with the arms and attached masses. Note that the angle ϕ is fixed.

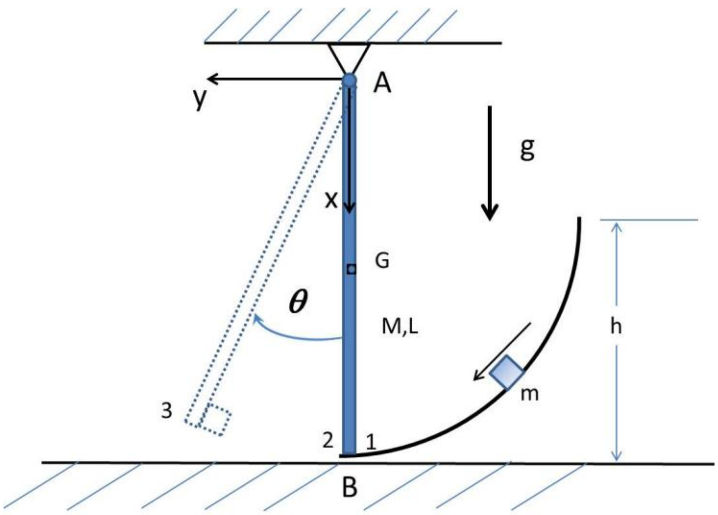


Find T and V, the kinetic and potential energy for this system.

Concept question: Is it possible to find the equation of motion of this system by requiring that:

$\frac{d}{dt}[T+V]=0$. a). Yes, b). No.

Problem 3: A particle of mass m slides down a frictionless surface. It then collides with and sticks to a uniform vertical rod of mass M and length L. Following the collision, the rod pivots about the point O. Point G is the mass center of the rod.

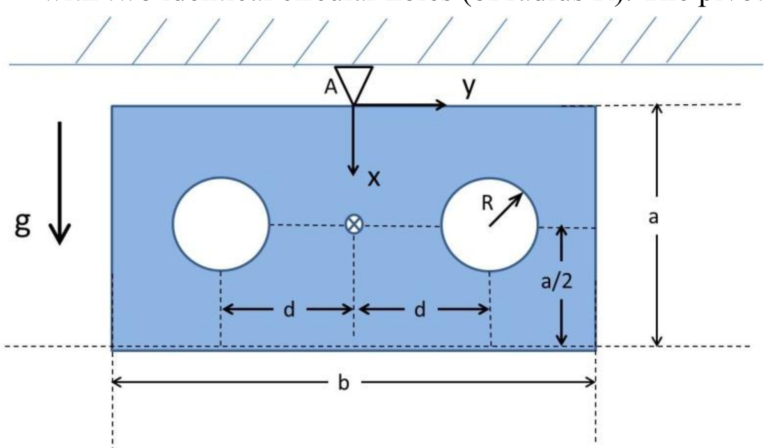


a). Find the kinetic energy, T, and the potential energy, V, of the system after the collision as a function of θ and $\dot{\theta}$.

Concept question: Angular momentum with respect to point A and (T+V) are both constant after the collision. A). True, B)False

Problem 4:

A pendulum consists of a rectangular plate (of thickness t) made of a material of density ρ , with two identical circular holes (of radius R). The pivot is at A.



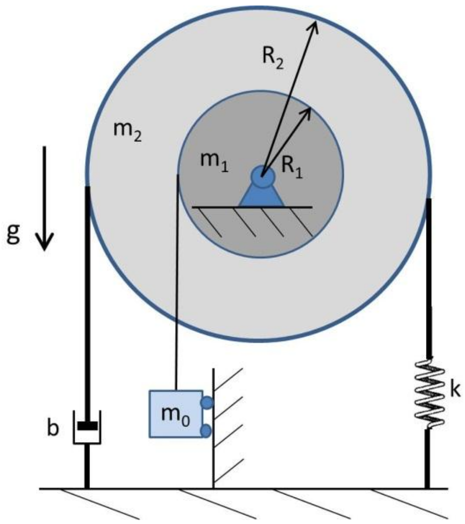
a). Find expressions for T and V, the kinetic and potential energies of the system in terms of the angle of rotation about point A.

b). This is a planar motion problem. How many degrees of freedom does this body have. What are the contraints on this rigid body.

Concept question: According to a strict definition of „translation": "All points on a rigid body must travel in parallel paths", does this rigid body exhibit rigid body translation?: a). Yes, b). No,

Problem 5:

Two uniform cylinders of mass m_1 and m_2 and radius R_1 and R_2 are welded together. This composite object rotates without friction about a fixed point. An inextensible massless string is wrapped without slipping around the larger cylinder. The two ends of the string are connected to the ground via, respectively, a spring of constant k and a dashpot of constant b. The smaller cylinder is connected to a block of mass m_o via an inextensible massless strap wrapped without slipping around the smaller cylinder. The block is constrained to move only vertically.

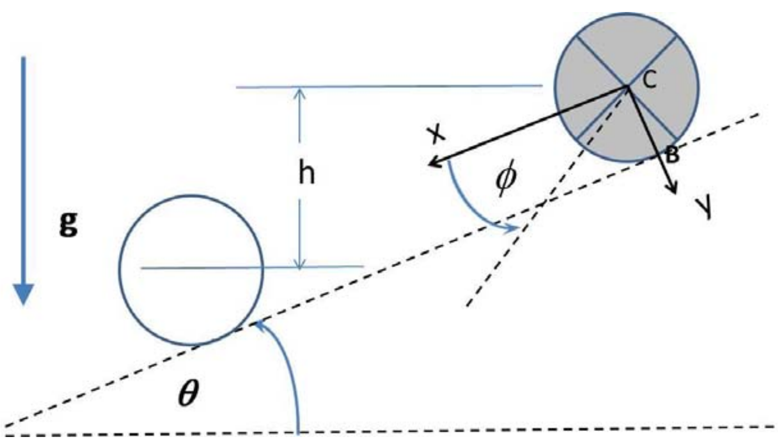


a) Find expressions for T and V the kinetic and potential energy of the system.

Concept question: For small values of the dashpot constant, b, if this single degree of freedom system is given an initial displacement from its static equilibrium position, will it exhibit oscillatory motion after release? a)Yes, b)No.

Problem 6:

A wheel is released at the top of a hill. It has a mass of 150 kg, a radius of 1.25 m, and a radius of gyration of k_G =0.6 m.

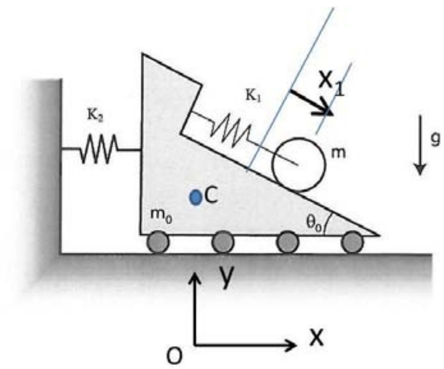


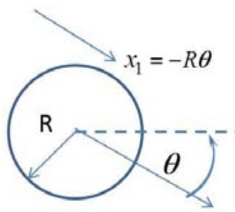
a). After release from the top of the hill the wheel rolls without slip down the hill. Find the kinetic energy, T, after the center of mass of the wheel has descended a vertical height „h".

b). Compute the ratio of the translational kinetic energy to the total kinetic energy of the system.

Concept question: If the wheel slips during its passage down the hill, is it correct to model it as a planar motion problem. A). Yes, b). No

Problem 7: The cart shown in the figure has mass m_o. It has an inclined surface as shown. A uniform disk of mass m, and radius R, rolls without slip on the inclined surface. The disk is restrained by a spring, K_1, attached at one end to the cart. The other end of the spring attaches to an axel passing through the center of the disk. The cart is restrained by a second spring, K_2, which is attached to a non-moving wall.





a). Find expressions for the kinetic energy, T, and potential energy, V, for the system.

b). Use Lagrange equations to find the equations of motion of the system.

Concept question: How many independent coordinates are required to account for the kinetic and potential energy in the system: a). 1, b). 2, c). 3.

Appendix: Computation of T and V for mechanical systems

In general the kinetic energy of a rigid body can be written as:

$$T = \frac{1}{2} m \vec{v}_{G/O} \cdot \vec{v}_{G/O} + \frac{1}{2} \vec{\omega} \cdot \vec{H}_{/G}, \qquad (1)$$

which is valid for 3D motion of any rigid body. In these expressions G refers to the center of mass of each rigid body. If the angular momentum can be expressed in terms of the mass moment of inertia matrix, computed with respect to the center of mass of the rigid body then equation (1) takes on the following form .

$$T = \frac{1}{2} m \vec{v}_{G/O} \cdot \vec{v}_{G/O} + \frac{1}{2} \vec{\omega} \cdot \left[\left[I_{/G} \right] \{\omega\} \right] \qquad (1a).$$

The expression $\left[I_{/G} \right] \{\omega\}$ is to be interpreted as the product of a scalar 3x3 matrix, $I_{/G}$, with a scalar, 3 element column vector $\{\omega\} = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$, where the entries are treated as scalars. The result of this matrix product is a column vector in which the elements are the three vector components of the angular momentum vector, $\left[I_{/G}\right]\{\omega\}=\{H_{/G}\}=\begin{Bmatrix} H_x\hat{i} \\ H_y\hat{j} \\ H_z\hat{k} \end{Bmatrix}$. Even though no unit vectors were used in taking the matrix product in equation (1a), by definition the result of computing this matrix product is assigned unit vectors as shown in the expression for $H_{/G}$. This approach of using what may seem to be an arbitrary definition is to avoid having to engage in the use of the mathematics of tensor notation.

Equation (1) makes use of vector dot products. This is intentional. The expressions $\vec{v}_{G/O}\cdot\vec{v}_{G/O}$ and $\vec{\omega}\cdot\vec{H}_{/G}$ are intended to be interpreted as true dot products between vectors with 3 components which include unit vectors, such as the expression for $H_{/G}$ given a few lines above, and where $\vec{\omega}=\begin{Bmatrix} \omega_x\hat{i} \\ \omega_y\hat{j} \\ \omega_z\hat{k} \end{Bmatrix}$.

Useful simplifications:

When the mass moment of inertia matrix has been computed with respect to **principal axes** then it is a diagonal matrix and equation (1a) takes on a simpler form:

$$T=\frac{1}{2}m\vec{v}_{G/O}\cdot\vec{v}_{G/O}+\frac{1}{2}\left[I_{xx}\omega_x^2+I_{yy}\omega_y^2+I_{zz}\omega_z^2\right] \tag{1b}$$

If the body rotates about a **fixed axis** passing through a point „A" then one may substitute the following formula:

$$T=\frac{1}{2}\vec{\omega}\cdot\vec{H}_{/A}=\frac{1}{2}\vec{\omega}\cdot\left[\left[I_{/A}\right]\{\omega\}\right] \tag{2}$$

As with equation (1b), when the mass moment of inertia matrix $I_{/A}$ has been computed with respect to principal axes, then it will be a diagonal matrix and equation (2) takes on the simpler form given in equation (2a)

$$T=\frac{1}{2}\left[I_{xx/A}\omega_x^2+I_{yy/A}\omega_y^2+I_{zz/A}\omega_z^2\right] \tag{2a}$$

In both equations (2) and (2a) the kinetic energy associated with translation has been absorbed into the computation using the mass moment of inertia matrix with respect to „A".

An even further simplification—single axis rotation about a principle axis:
When the rotation is about a single fixed axis passing through a point „A" and that axis is a

principal axis then equations (2) and (2a) take on an especially simple form.

$$T = \frac{1}{2}\vec{\omega}\cdot\left[\left[I_{/A}\right]\{\omega\}\right] = \frac{1}{2} I_{ii/A}\omega_i^2 \text{ where i is the axis of rotation.} \qquad (2b)$$

Planar motion:

When the motion of a rigid body is confined to translation in a plane, for example the x-y plane, and allowed to rotate only about a principal axis, which is perpendicular to that plane, the resulting motion is defined as planar motion. In such cases the axis of rotation is allowed to move, as when a wheel rolls down a hill. For such problems the kinetic energy expression becomes:

$$T = \frac{1}{2} m \vec{v}_{G/O}\cdot\vec{v}_{G/O} + \frac{1}{2} I_{ii/G}\omega_i^2 \qquad (3)$$

When the plane is the x-y plane then the axis „i" in equation (3) is the z axis.

Potential energy computations for mechanical systems:

Potential energy V is defined as the negative of the work done by the external conservative forces.

$V = -\int \vec{F}_{conservative}\cdot d\vec{r}$. For purely mechanical systems the only potential(conservative) forces are due to gravity and springs. Once one knows the standard ways of computing potential energy for gravity and springs the work is very straightforward. However, clever choices in picking the reference potential energy levels and coordinate system can make a big difference in the ease of implementation of the equations of motion that result. This brief write-up introduces that discussion in one way.

When both gravity and springs are present in a problem, the system will sometimes have a stable, non-moving, static equilibrium state. Very often this is the result of a static deflection of the spring which produces a force which balances the weight of the object. An example is a mass hanging from a spring. The dynamic motions of such systems often occur around the static equilibrium position of the system. It is often advantageous to choose coordinates such that they are zero when in the static equilibrium position. Computation of the potential energy function, V, when using coordinates measured with respect to static equilibrium positions requires some care, but pays off later on. This will be done by example in this problem set.

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