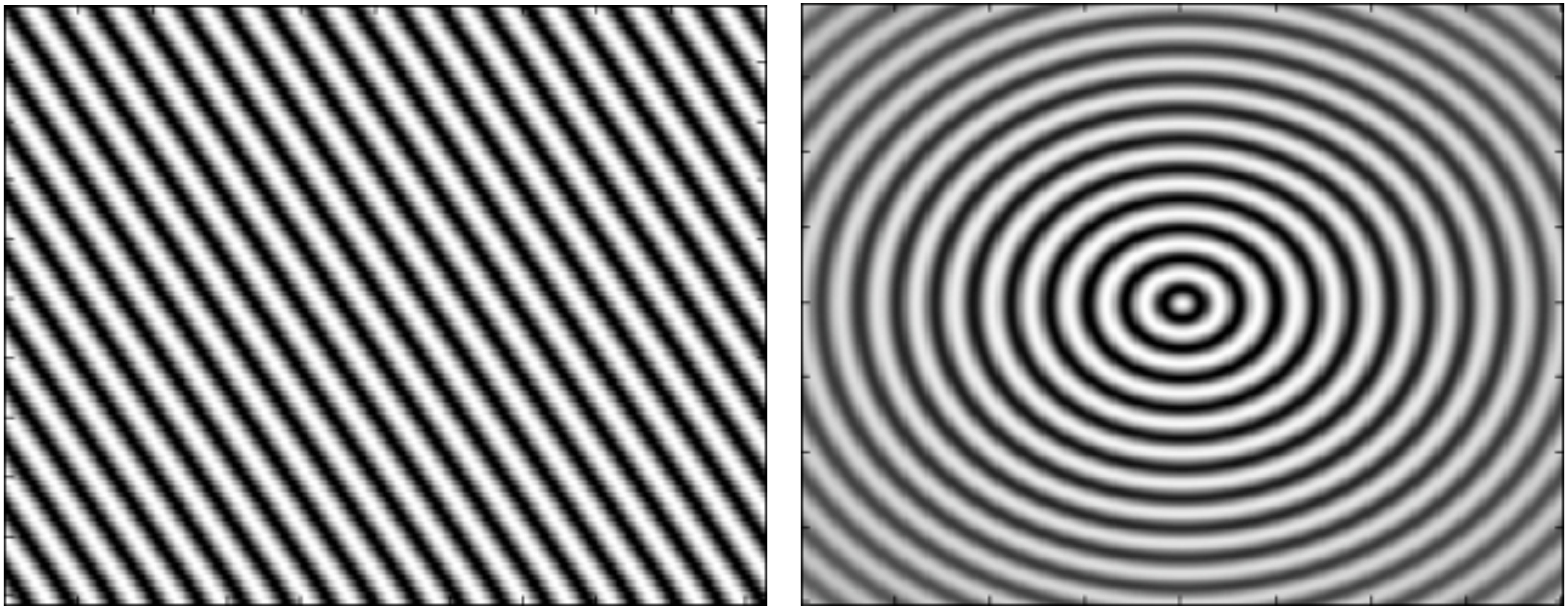
The 3D wave equation

In three-dimensions, the Wave Equation is generalized as

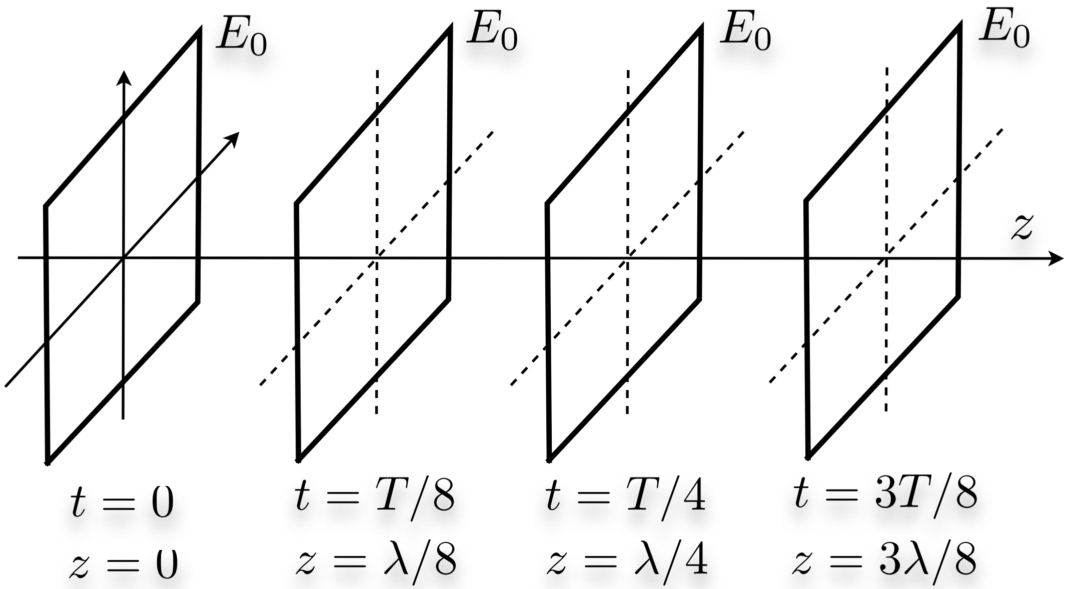
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0.$$

Our familiar plane and spherical waves are special solutions.



Plane wave   Spherical wave

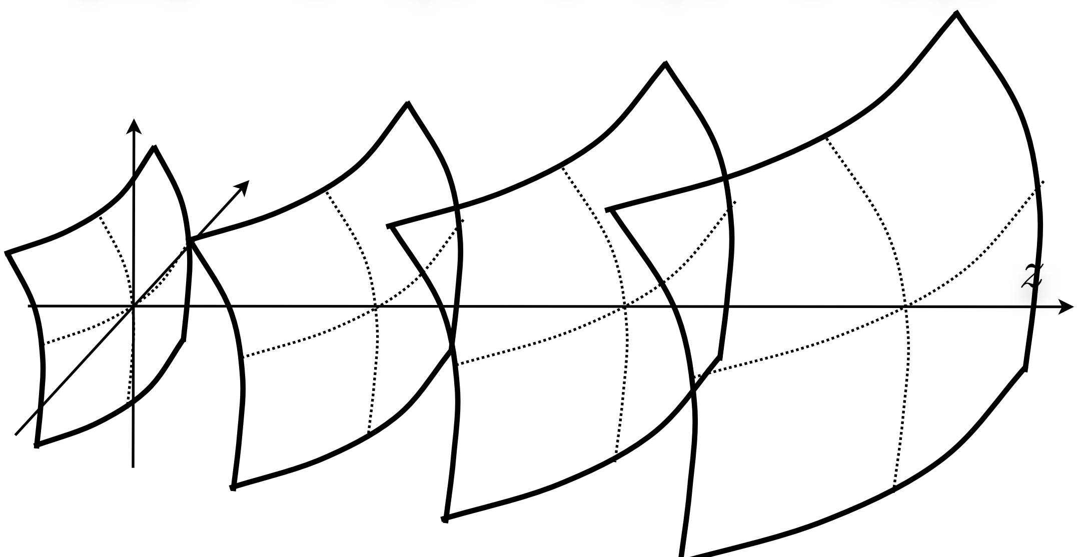
Planar and Spherical Wavefronts



Planar wavefront (plane wave):

The wave phase is constant along a planar surface (the wavefront).

As time evolves, the wavefronts propagate at the wave speed without changing; we say that the wavefronts are *invariant to propagation* in this case.

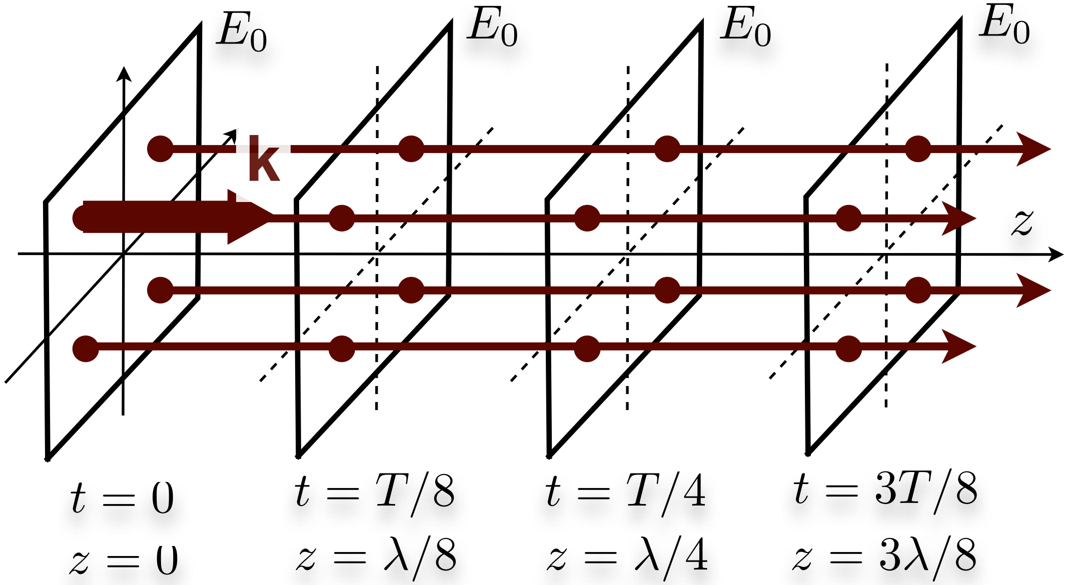


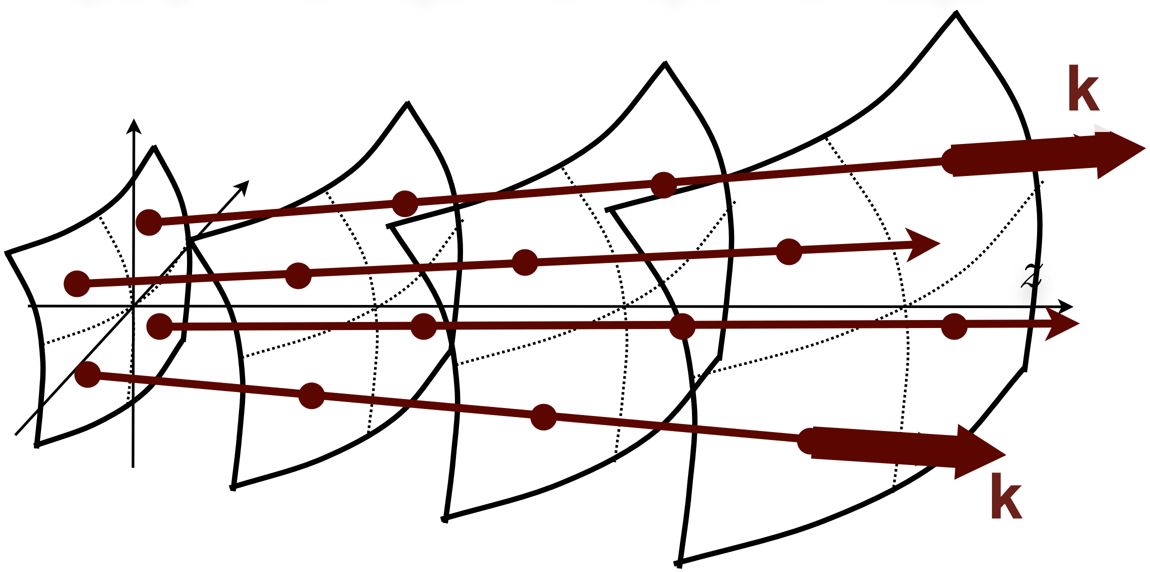
Spherical wavefront (spherical wave):

The wave phase is constant along a spherical surface (the wavefront).

As time evolves, the wavefronts propagate at the wave speed and expand outwards while preserving the wave's energy.

Wavefronts, rays, and wave vectors





Rays are:

1) normals to the wavefront surfaces
2) trajectories of "particles of light"

Wave vectors:

At each point on the wavefront, we may assign a normal vector **k**

This is known as the wave vector; it magnitude *k* is the wave number and it is defined as

$$k \equiv |\mathbf{k}| = \frac{2\pi}{\lambda}$$

3D wave vector from the wave equation

We try a sinusoidal solution

$$a \exp\left\{i\left(k_x x + k_y y + k_z z - \omega t\right)\right\} =$$

$$= a \exp\left\{i\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right\}, \qquad \text{where}$$

$\mathbf{k} = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y + \hat{\mathbf{z}}k_z$ is the wave vector, and

$\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$ is the Cartesian

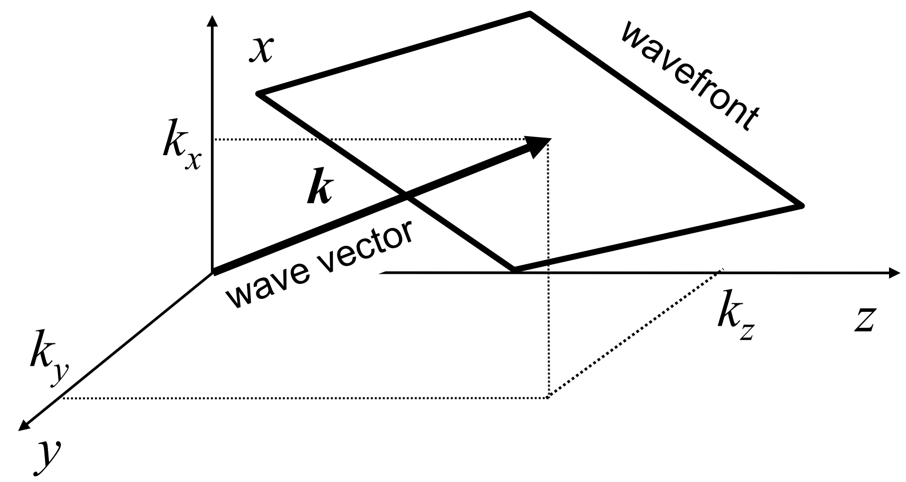
coordinate vector, to the 3D wave equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0 \Rightarrow$$

$$-a\left(k_x^2 + k_y^2 + k_z^2 - \frac{\omega^2}{c^2}\right)\mathrm{e}^{i(k_x x + k_y y + k_z z - \omega t)} = 0 \Rightarrow$$

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}.$$

That is, $|\mathbf{k}| = \frac{\omega}{c} = \frac{2\pi n}{\lambda} \equiv k$ (wave number.)



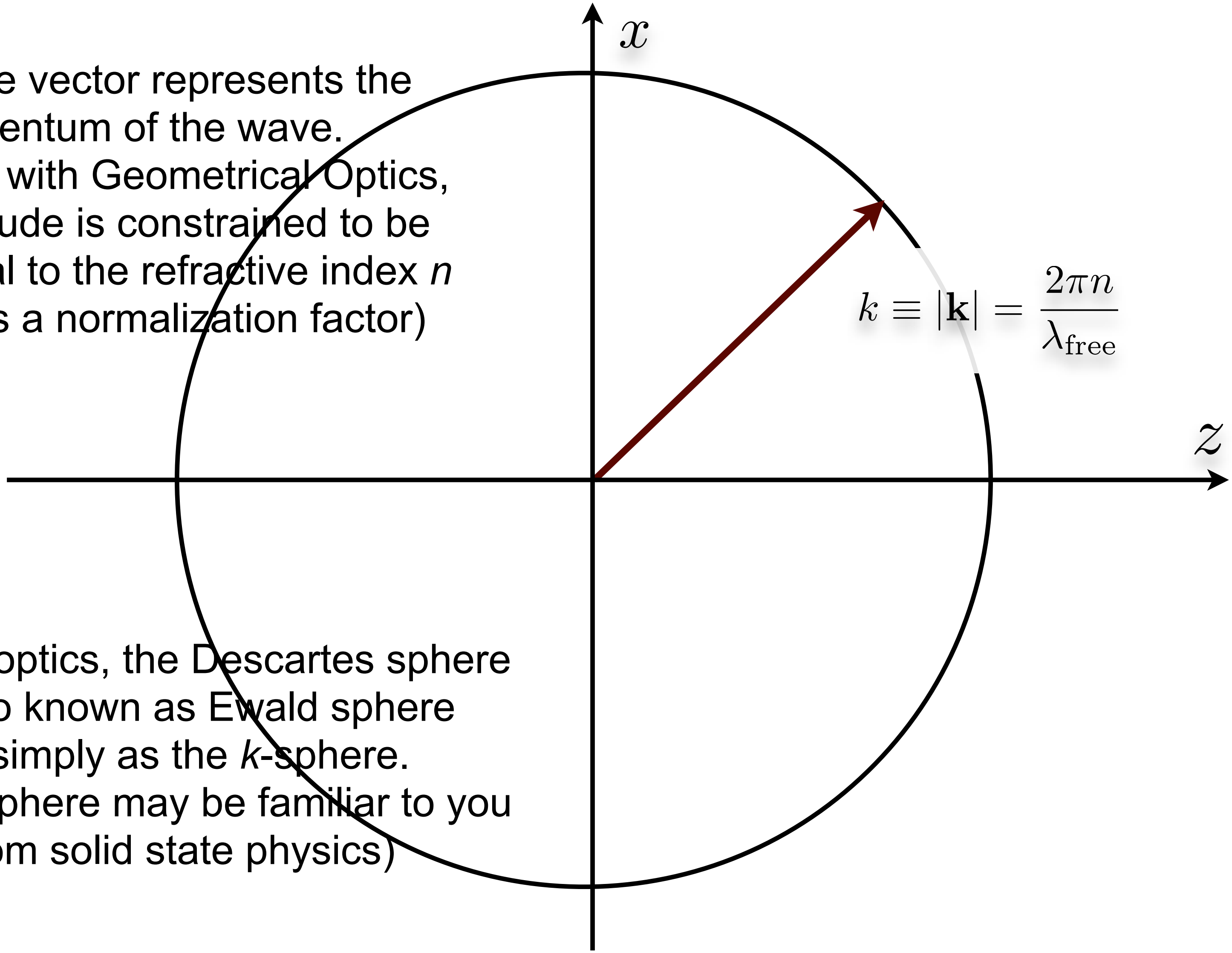
The wavefront is the surface

$$\mathbf{k}\cdot\mathbf{r} = \text{const.}$$

i.e., the locus of points on the wave that have the same phase (modulo 2π) after propagating by the same time t.

3D wave vector and the Descartes sphere

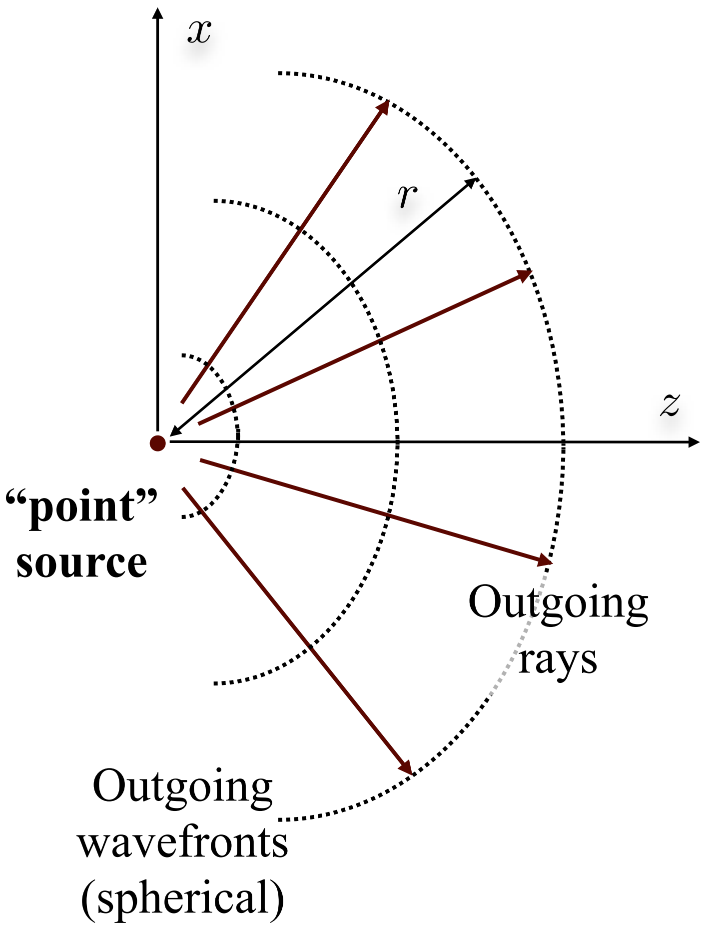
The wave vector represents the momentum of the wave. Consistent with Geometrical Optics, its magnitude is constrained to be proportional to the refractive index *n* ($2\pi/\lambda_{\text{free}}$ is a normalization factor)



$$k \equiv |\mathbf{k}| = \frac{2\pi n}{\lambda_{\text{free}}}$$

In wave optics, the Descartes sphere is also known as Ewald sphere or simply as the *k*-sphere. (Ewald sphere may be familiar to you from solid state physics)

Spherical wave



The wavefront in this case is a sphere

$$kr = \text{const.}, \qquad \text{where} \quad r \equiv |\mathbf{r}|\,.$$

Without proof (pls. see the textbook) we assert

$$f(\mathbf{r},t) = a\frac{\cos\left(kr - \omega t - \pi/2\right)}{r}$$

In complex representation,

$$\hat{f}(\mathbf{r},t) = a\frac{\exp\left\{i\left(kr - \omega t\right)\right\}}{ir},$$

and in phasor notation (dropping the $\mathrm{e}^{-i\omega t}$)

$$\hat{f}(\mathbf{r}) = \frac{a}{ir}\exp\left\{ikr\right\}.$$

In the paraxial approximation, $z \gg |x|\,,|y|$ so

$$r = \sqrt{x^2+y^2+z^2} = z\sqrt{1+\frac{x^2+y^2}{z^2}} \approx z + \frac{x^2+y^2}{2z} \Rightarrow$$

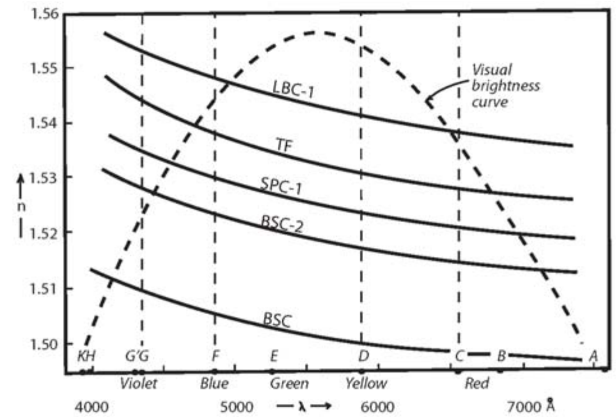
$$\begin{aligned}\hat{f}(\mathbf{r}) &\approx \frac{a}{iz}\exp\left\{ikz\right\}\exp\left\{ik\frac{x^2+y^2}{2z}\right\} \\ &= \frac{a}{iz}\exp\left\{i\frac{2\pi}{\lambda}z\right\}\exp\left\{i\pi\frac{x^2+y^2}{\lambda z}\right\}.\end{aligned}$$

Dispersive waves

We have learnt from Geometrical Optics that the speed of light can be wavelength dependent, e.g. due to material dispersion $n(\lambda)$. This means that the wave equation for light waves must be written as

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{c^2(k)} \frac{\partial^2 f}{\partial t^2},$$

where $c(k) = c_{\text{vacuum}}/n(k)$ denotes the dependence of c on the wave number $k = 2\pi/\lambda$. This kind of wave is called **dispersive.**

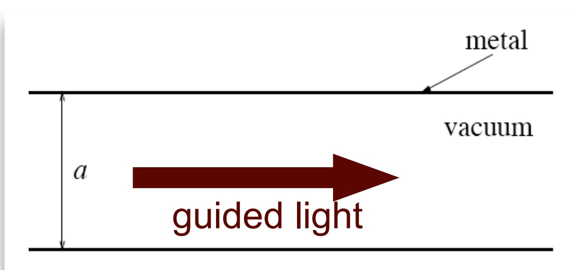


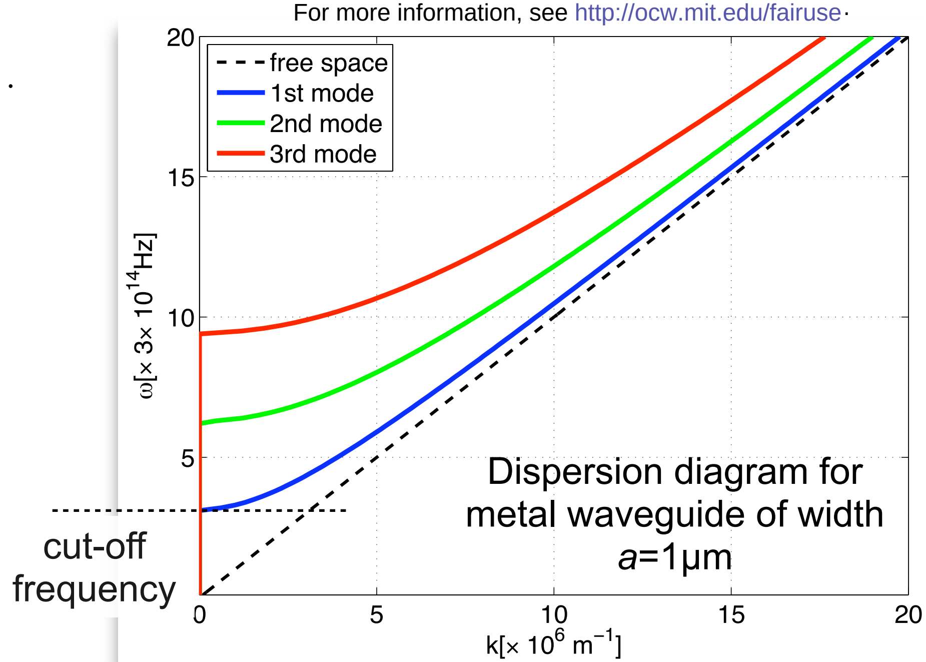
Dispersion curves for glass

Fig. 9X,Y in Jenkins, Francis A., and Harvey E. White. *Fundamentals of Optics*. 4th ed. New York, NY: McGraw-Hill, 1976. ISBN: 9780070323308.

Another example of a dispersive wave is a **guided** wave. It turns out that, due to the boundary conditions at the waveguide's edge, the simple dispersion relation $c = \lambda\nu$ does **not** hold for a waveguide, and it must be replaced by a different relationship. Without going into the details here, the dispersion relationship for the metal waveguide shown on the left is

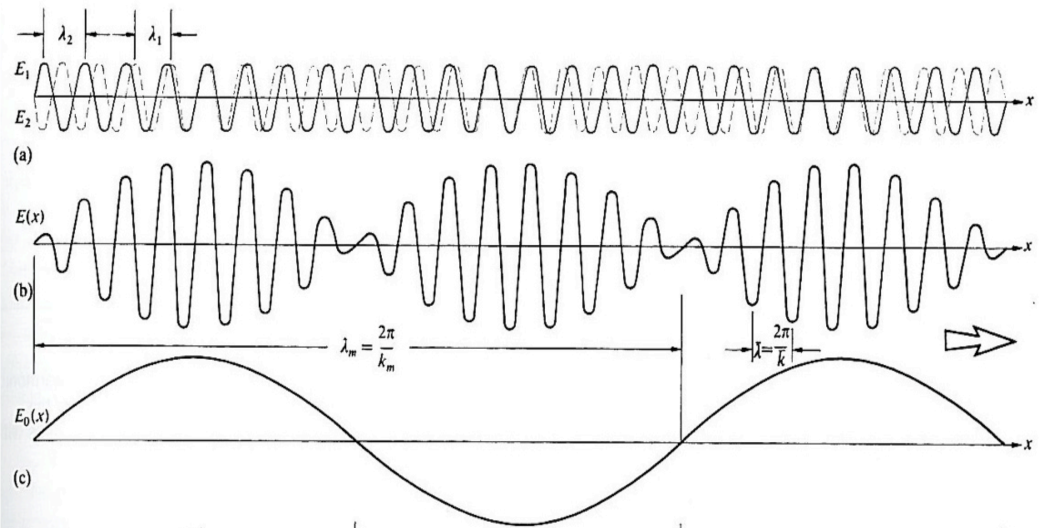
$$\left(\frac{m\pi}{a}\right)^2 + k^2 = \left(\frac{\omega}{c}\right)^2, \qquad m = 0, \pm 1, \pm 2, \ldots$$

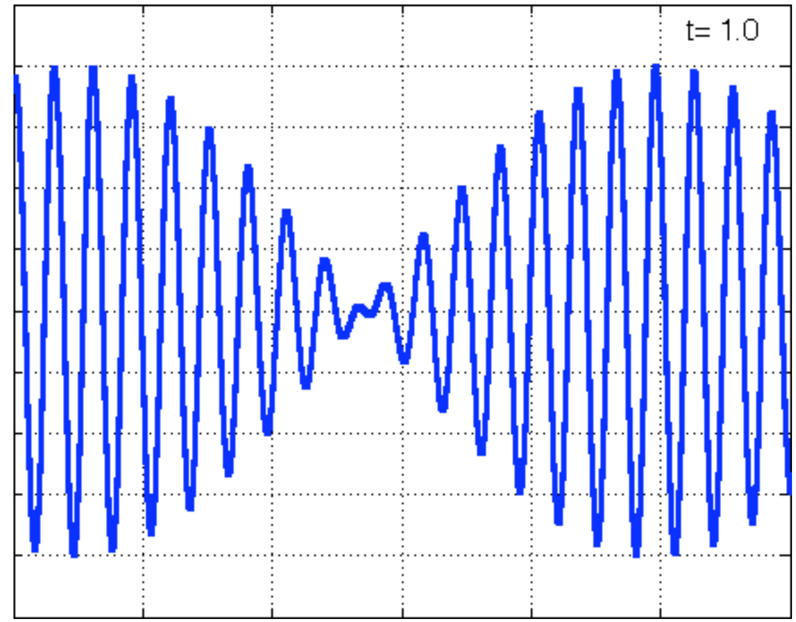




Dispersion diagram for metal waveguide of width *a*=1μm

Superposition of waves at different frequencies





Consider two waves of different frequency and wavelength

$$f_1(z,t) = a\cos\left(k_1 z - \omega_1 t\right), \qquad f_2(z,t) = a\cos\left(k_2 z - \omega_2 t\right).$$

Their superposition is

$$\begin{aligned} f(z,t) &= f_1(z,t) + f_2(z,t) \\ &= a\cos\left(k_1 z - \omega_1 t\right) + a\cos\left(k_2 z - \omega_2 t\right) \\ &= 2a\cos\frac{(k_1+k_2)\,z - (\omega_1+\omega_2)\,t}{2}\cos\frac{(k_1-k_2)\,z - (\omega_1-\omega_2)\,t}{2} \\ &\equiv 2a\cos\left(k_\mathrm{c} z - \omega_\mathrm{c} t\right)\cos\left(k_\mathrm{m} z - \omega_\mathrm{m} t\right), \end{aligned}$$

where $k_\mathrm{c} \equiv \dfrac{k_1+k_2}{2}$ and $\omega_\mathrm{c} \equiv \dfrac{\omega_1+\omega_2}{2}$;

are the wave vector and frequency of the **carrier** wave; and a

$$k_\mathrm{m} \equiv \frac{k_1-k_2}{2} \quad \text{and} \quad \omega_\mathrm{m} \equiv \frac{\omega_1-\omega_2}{2};$$

are the wave vector and frequency of the **modulation.** ω_m is also referred to as **beat frequency.**

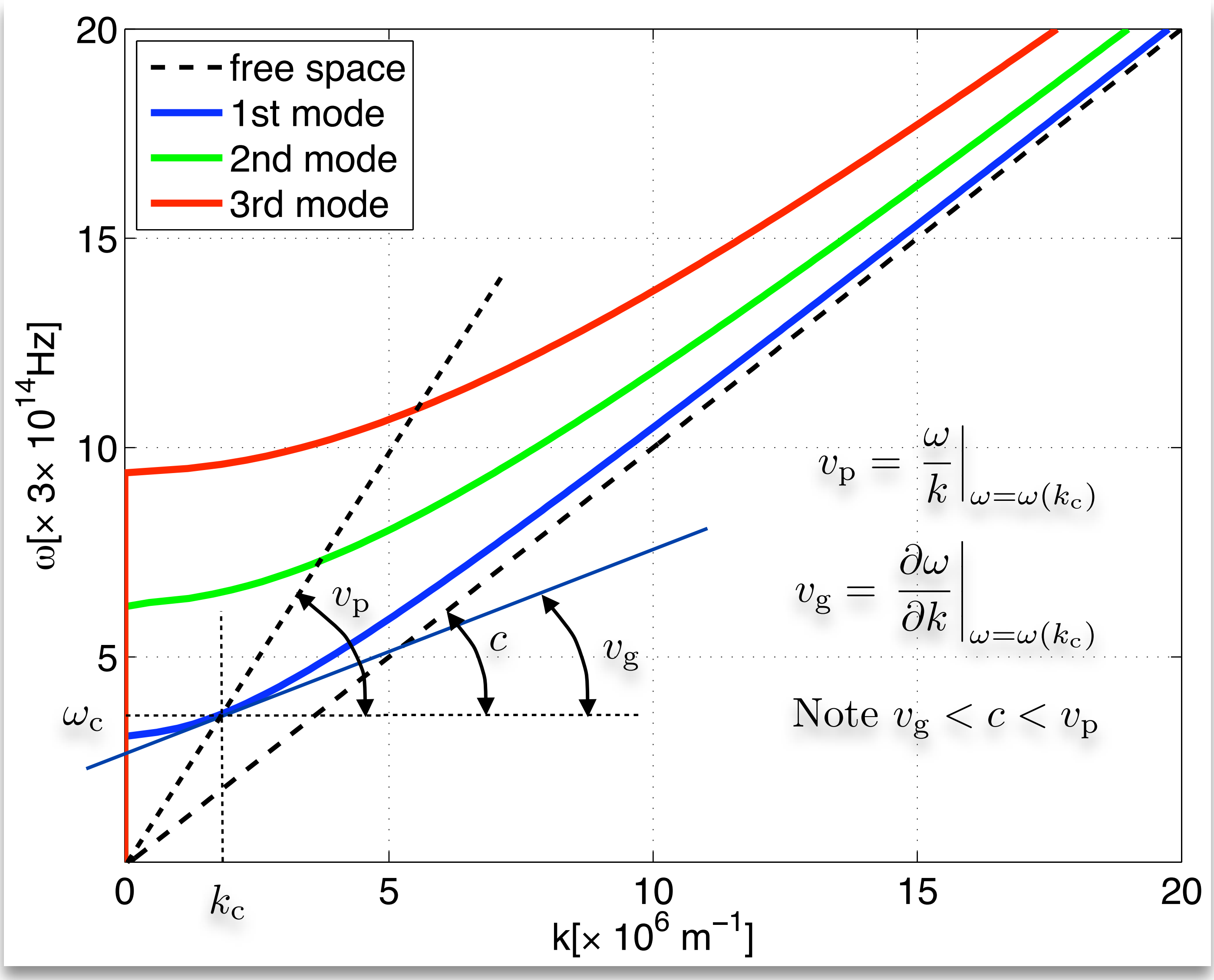
If $\omega_1 \approx \omega_2$ and $k_1 \approx k_2$, then

$$v_\mathrm{p} = \frac{\omega}{k}, \quad \text{and} \quad v_\mathrm{g} = \left.\frac{\partial\omega}{\partial k}\right|_{\omega=\omega(k)}.$$

The carrier wave propagates at the **phase velocity** $v_\mathrm{p} \equiv \dfrac{\omega_\mathrm{c}}{k_\mathrm{c}}$,

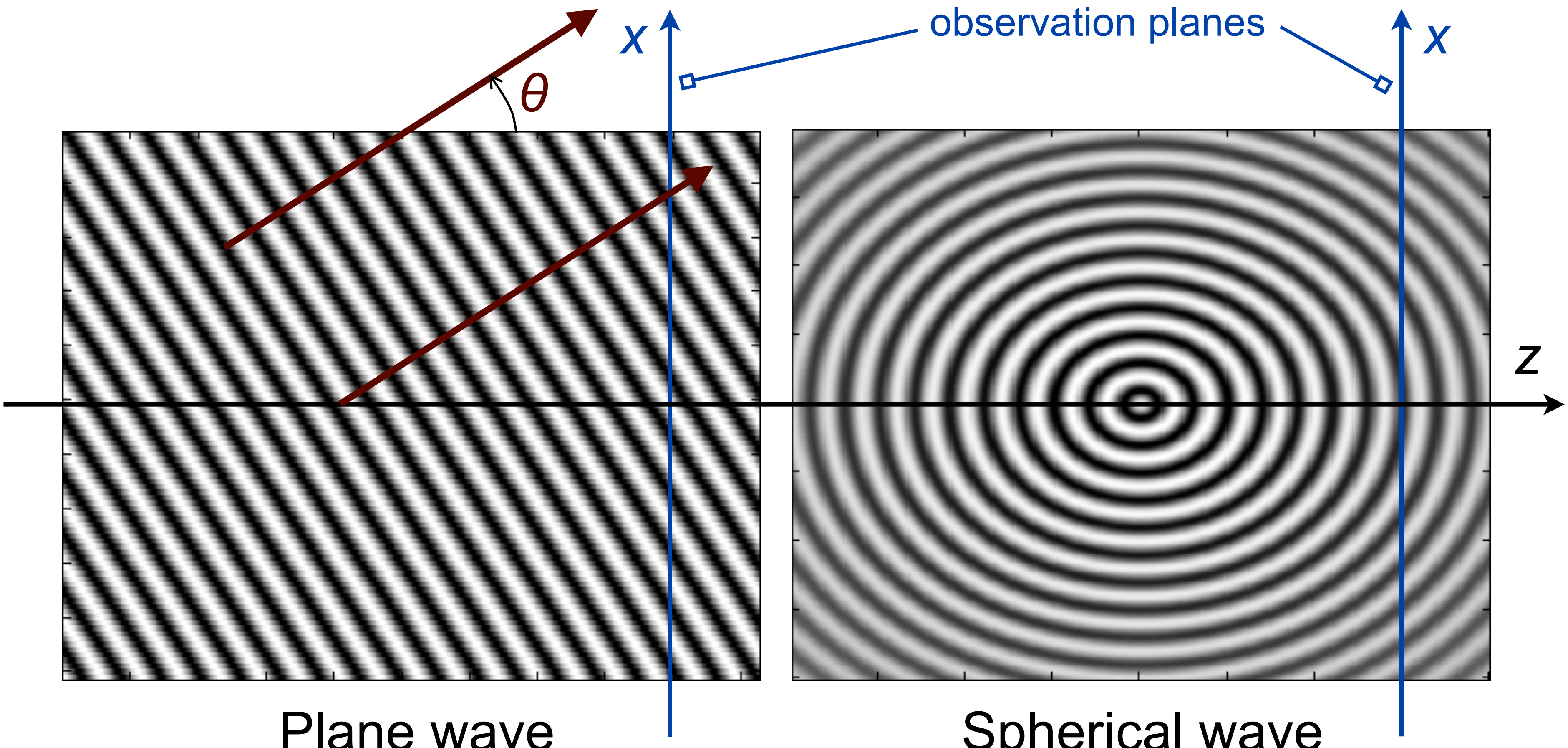
whereas the modulation propagates at the **group velocity** $v_\mathrm{g} \equiv \dfrac{\omega_\mathrm{m}}{k_\mathrm{m}}$.

Group and phase velocity



Spatial frequencies

We now turn to a monochromatic (single color) optical field.
The field is often observed (or measured) at a planar surface along the optical axis z.
The wavefront shape at the observation plane is, therefore, of particular interest.



Plane wave

Spherical wave

Spatial frequencies

$$E(x,z) = \exp\left\{ i\frac{2\pi}{\lambda}\left(x\sin\theta + z\cos\theta\right)\right\}$$
$$= \exp\left\{ i2\pi\left(\frac{x}{\Lambda} + \text{const}\right)\right\}$$

$$f_X \equiv \frac{1}{\Lambda} = \frac{\lambda}{\sin\theta}$$

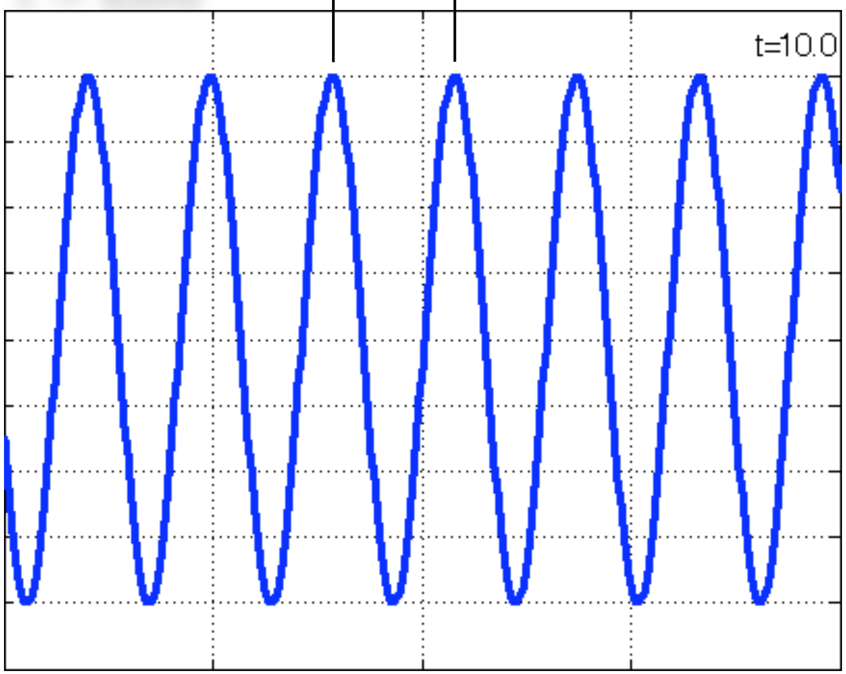
spatial frequency

$$E(x,z) \sim \exp\left\{ i\frac{2\pi}{\lambda}\sqrt{x^2+y^2+z^2}\right\}$$
$$\approx \exp\left\{ i\frac{2\pi z}{\lambda}\right\}\exp\left\{ i\pi\frac{x^2+y^2}{\lambda z}\right\}$$

$$\Lambda \equiv \frac{\lambda}{\sin\theta}$$

Chirped frequency $f_X(x) \sim \frac{2x}{\lambda z}$

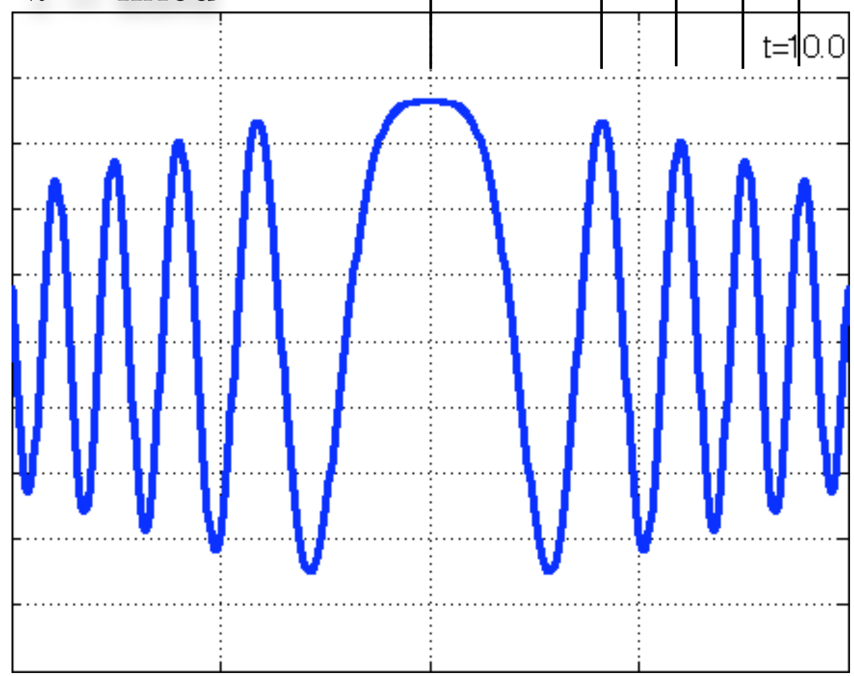
$z =$ fixed



x

Plane wave

$z =$ fixed



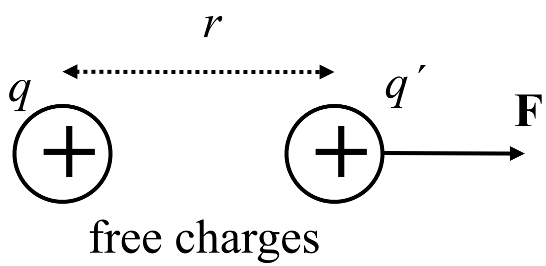
x

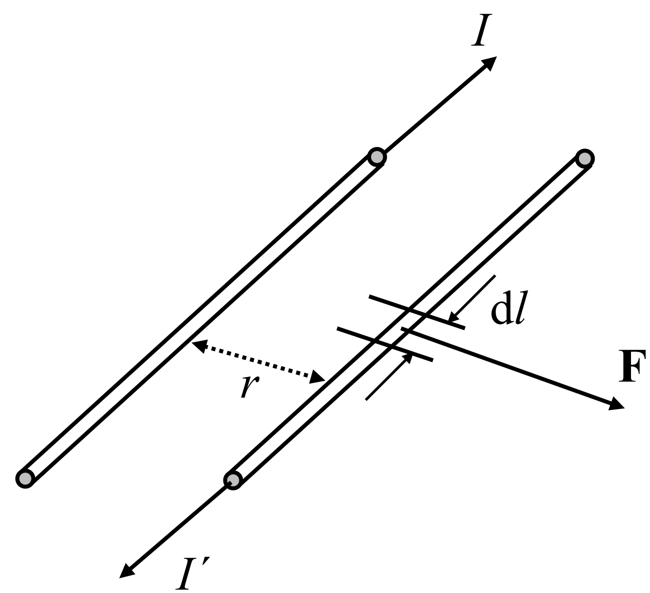
Spherical wave

Today

- Electromagnetics
  - Electric (Coulomb) and magnetic forces
  - Gauss Law: electrical
  - Gauss Law: magnetic
  - Faraday's Law
  - Ampère-Maxwell Law
  - Maxwell's equations ⇒ Wave equation
  - Energy propagation
    - Poynting vector
    - average Poynting vector: intensity
  - Calculation of the intensity from phasors
    - Intensity

Electric and magnetic forces





Coulomb force

$$|\mathbf{F}| = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{r^2}$$

(dielectric) permitivity of free space

$$\epsilon_0 = 8.8542 \times 10^{-12}\,\frac{\mathrm{Cb}^2}{\mathrm{N}\cdot\mathrm{m}^2}$$

Magnetic force

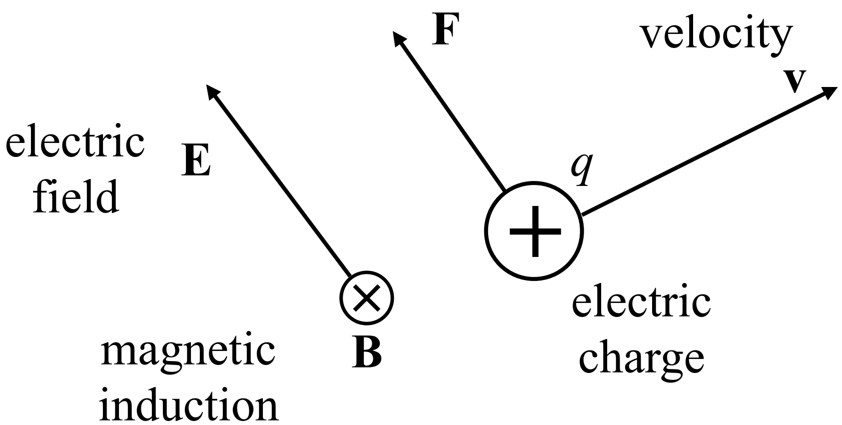
$$\frac{\mathrm{d}|\mathbf{F}|}{\mathrm{d}l} = \mu_0\,\frac{II'}{2\pi r}$$

(magnetic) permeability of free space

$$\mu_0 = 4\pi \times 10^{-7}\,\frac{\mathrm{N}\cdot\sec^2}{\mathrm{Cb}^2}$$

Electric and magnetic fields

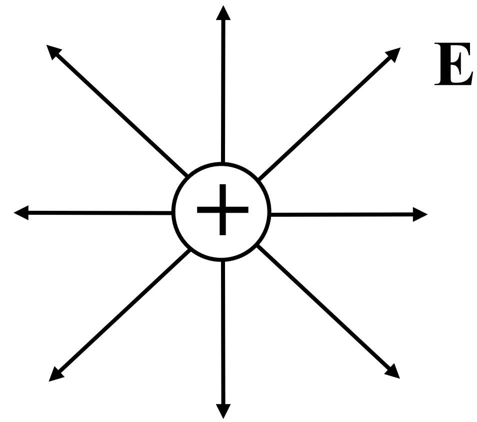
<u>Observation</u>



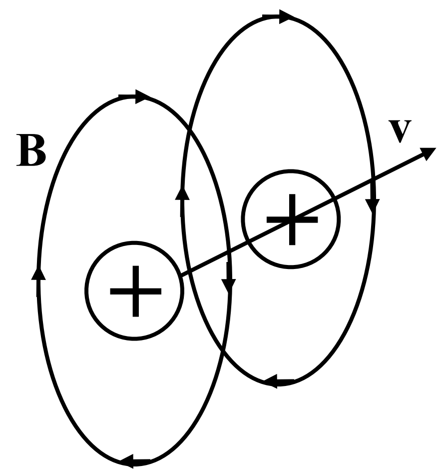
$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Lorentz force

<u>Generation</u>

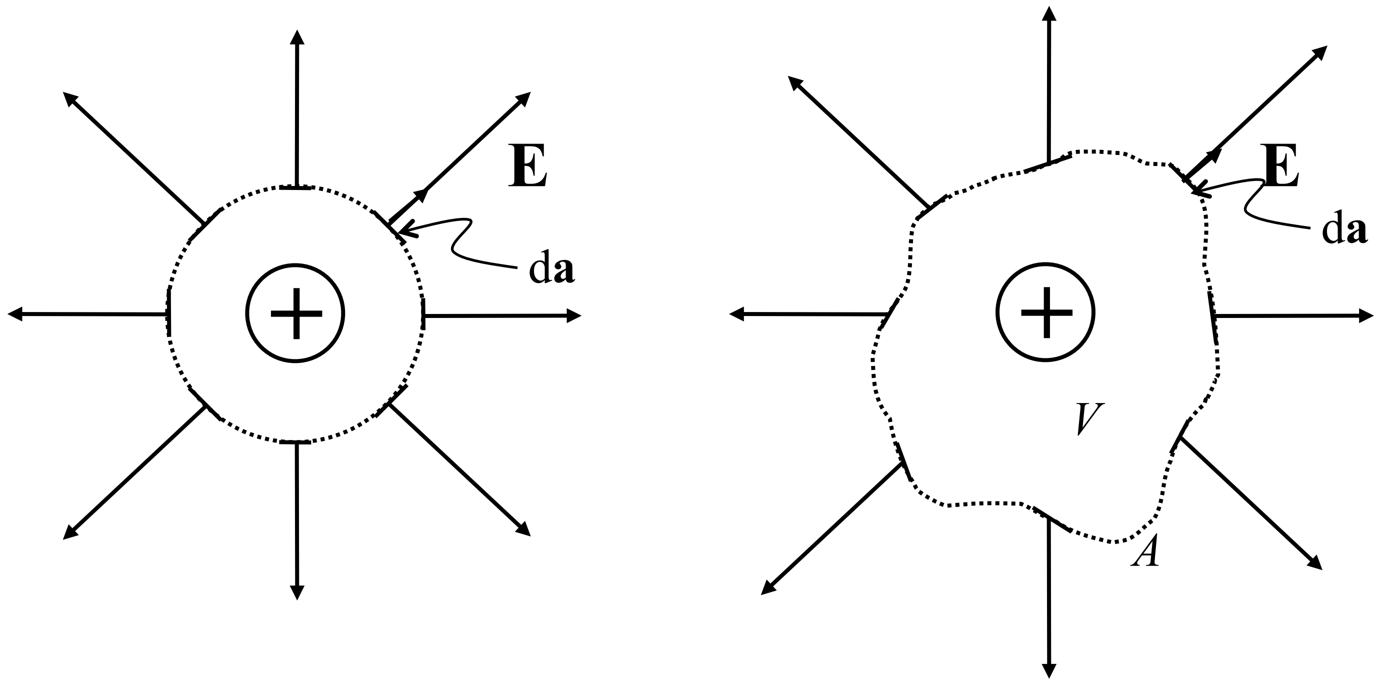


static charge:
⇒
electric field



moving charge
(electric current):
⇒
magnetic field

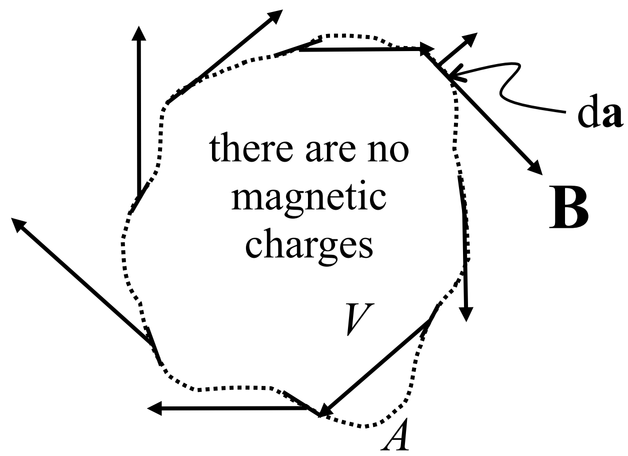
Gauss Law: electric field



$$\oiint_A \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\epsilon_0} \iiint_V \rho \mathrm{d}V \quad \overset{\text{Gauss theorem}}{\Longleftrightarrow} \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

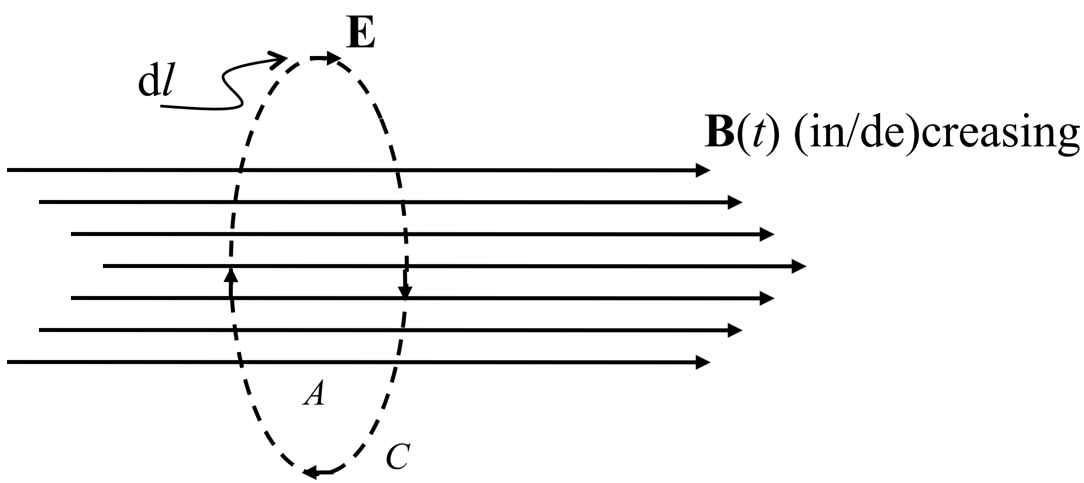
ρ: charge density

Gauss Law: magnetic field



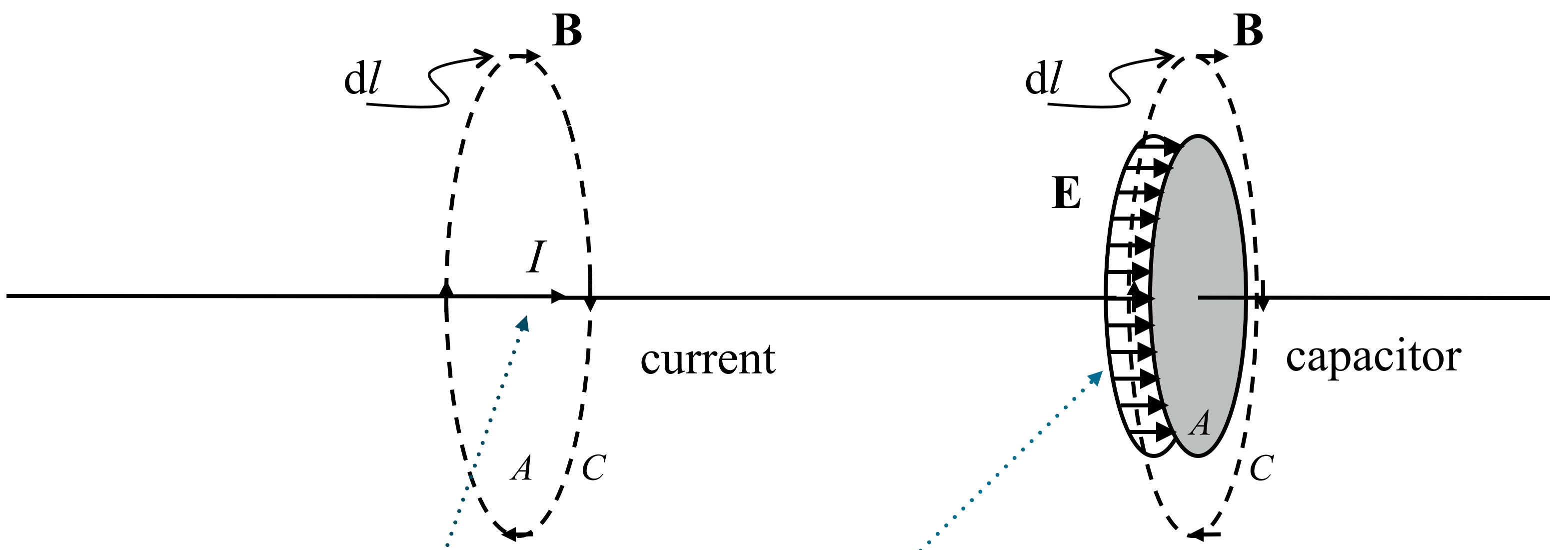
$$\oiint_A \mathbf{B} \cdot \mathrm{d}\mathbf{a} = 0 \quad \overset{\text{Gauss theorem}}{\Longleftrightarrow} \quad \nabla \cdot \mathbf{B} = 0$$

Faraday's Law: electromotive force



$$\oint_C \mathbf{E}\cdot \mathrm{d}l = -\frac{\mathrm{d}}{\mathrm{d}t}\iint_A \mathbf{B}\cdot \mathrm{d}\mathbf{a} \quad \overset{\text{Stokes theorem}}{\Longleftrightarrow} \quad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Ampère-Maxwell Law: magnetic induction



$$\oint_C \mathbf{B} \cdot \mathrm{d}l = \mu_0 \left(\iint_A \mathbf{J} \cdot \mathrm{d}\mathbf{a} + \epsilon_0 \iint_A \frac{\partial \mathbf{E}}{\partial t} \cdot \mathrm{d}\mathbf{a} \right) \overset{\text{Stokes theorem}}{\Longleftrightarrow} \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Maxwell's Equations (free space)

Integral form

$$\oiint_A \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\epsilon_0} \iiint_V \rho \mathrm{d}V$$

$$\oiint_A \mathbf{B} \cdot \mathrm{d}\mathbf{a} = 0$$

$$\oint_C \mathbf{E} \cdot \mathrm{d}l = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_A \mathbf{B} \cdot \mathrm{d}\mathbf{a}$$

$$\oint_C \mathbf{B} \cdot \mathrm{d}l = \mu_0 \left(\iint_A \mathbf{J} \cdot \mathrm{d}\mathbf{a} + \epsilon_0 \iint_A \frac{\partial \mathbf{E}}{\partial t} \cdot \mathrm{d}\mathbf{a} \right)$$

Differential form

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Wave Equation for electromagnetic waves

We will derive the Wave Equation
from Maxwell's electromagnetic equations
in free space and in the absence of charges and currents.
Starting from Faraday's equation,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial (\nabla \times \mathbf{B})}{\partial t}$$

Now we substitute Ampère–Maxwell's Law

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

the following identity from vector calculus

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E},$$

and Gauss' Law for electric fields,

$$\nabla \cdot \mathbf{E} = 0.$$

Collecting all these results, we obtain

$$\nabla^2 \mathbf{E} - \mu_o \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.$$

Comparing with the 3D Wave Equation,

$$\nabla^2 f - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = 0,$$

we see that *each component* of the vector $\mathbf{E}$
satisfies the Wave Equation with velocity

$$\frac{1}{c^2} = \mu_o \epsilon_0 \Rightarrow c = \frac{1}{\sqrt{\mu_o \epsilon_0}}$$

Since $\epsilon_0 = 8.8542 \times 10^{-12} \mathrm{Cb}^2/\mathrm{N} \cdot \mathrm{m}^2$,
$\mu_0 = 4\pi \times 10^{-7} \mathrm{N} \cdot \mathrm{sec}^2/\mathrm{Cb}^2$, we obtain
the speed of electromagnetic waves in vacuum

$$c = 3 \times 10^8 \, \frac{\mathrm{m}}{\mathrm{sec}}.$$

2.71 / 2.710 Optics
Spring 2009