Lecture 19: Dynamic Programming I: Memoization, Fibonacci, Shortest Paths, Guessing

Lecture Overview

- Memoization and subproblems
- Examples
  - Fibonacci
  - Shortest Paths
- Guessing & DAG View

Dynamic Programming (DP)

Big idea, hard, yet simple

- Powerful algorithmic design technique
- Large class of seemingly exponential problems have a polynomial solution ("only") via DP
- Particularly for optimization problems (min / max) (e.g., shortest paths)

* DP $\approx$ "controlled brute force"
* DP $\approx$ recursion + re-use

History

Richard E. Bellman (1920-1984)
Richard Bellman received the IEEE Medal of Honor, 1979. "Bellman . . . explained that he invented the name 'dynamid programming' to hide the fact that he was doing mathematical research at RAND under a Secretary of Defense who 'had a pathological fear and hatred of the term, research'. He settled on the term 'dynamic programming' because it would be difficult to give a 'pejorative meaning' and because 'it was something not even a Congressman could object to' " [John Rust 2006]

Fibonacci Numbers

$$F_1 = F_2 = 1; \quad F_n = F_{n-1} + F_{n-2}$$

Goal: compute F_n

Naive Algorithm

follow recursive definition

```
fib($n$):
        if $n \leq 2$: return $f = 1$
        else: return $f = \text{fib}(n-1) + \text{fib}(n-2)$
$\implies T(n) = T(n-1) + T(n-2) + O(1) \geq F_n \approx \varphi^n$
        $\geq 2T(n-2) + O(1) \geq 2^{n/2}$
        EXPONENTIAL — BAD!
```

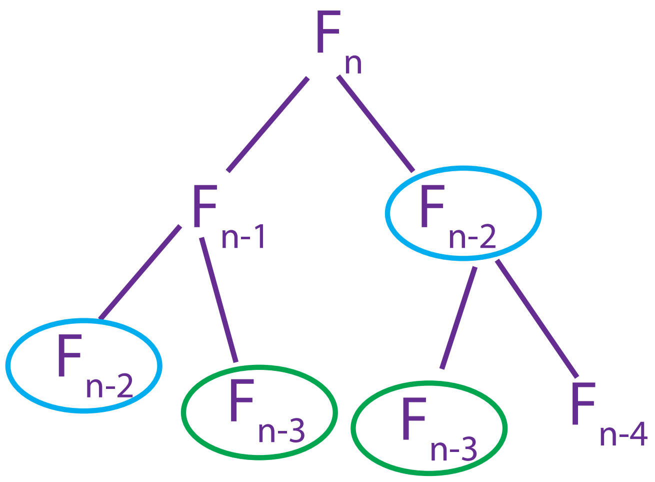


Figure 1: Naive Fibonacci Algorithm.

Memoized DP Algorithm

Remember, remember

```
memo = { }
fib($n$):
        if $n$ in memo: return memo[$n$]
        else: if $n \leq 2$ : $f = 1$
              else: $f =$ fib$(n-1)$ + fib$(n-2)$
              memo[$n$] = $f$
              return $f$
```

- $\implies$ fib(k) only recurses first time called, $\forall k$
- $\implies$ only n nonmemoized calls: $k = n, n-1, \ldots, 1$
- memoized calls free ($\Theta(1)$ time)
- $\implies$ $\Theta(1)$ time per call (ignoring recursion)

POLYNOMIAL — GOOD!

* DP $\approx$ recursion + memoization

- memoize (remember) & re-use solutions to subproblems that help solve problem
  - in Fibonacci, subproblems are $F_1, F_2, \ldots, F_n$

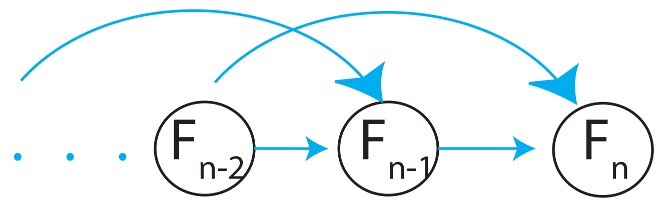
* $\implies$ time = # of subproblems · time/subproblem

- Fibonacci: # of subproblems is n, and time/subproblem is $\Theta(1) = \Theta(n)$ (ignore recursion!).

Bottom-up DP Algorithm

```
fib = {}                                              } Θ(n)
for $k$ in [1, 2, ..., $n$]:
    if $k \leq 2$: $f = 1$                  } Θ(1)
    else: $f$ = fib[$k-1$] + fib[$k-2$]
    fib[$k$] = $f$
return fib[$n$]
```

- exactly the same computation as memoized DP (recursion "unrolled")
- in general: topological sort of subproblem dependency DAG



- practically faster: no recursion
- analysis more obvious
- can save space: just remember last 2 fibs $\implies \Theta(1)$

[Sidenote: There is also an $O(\lg n)$-time algorithm for Fibonacci, via different techniques]

Shortest Paths

- Recursive formulation:
  $\delta(s,v) = \min\{w(u,v) + \delta(s,u) \big| (u,v) \in E\}$
- Memoized DP algorithm: takes infinite time if cycles!
  in some sense necessary to handle negative cycles

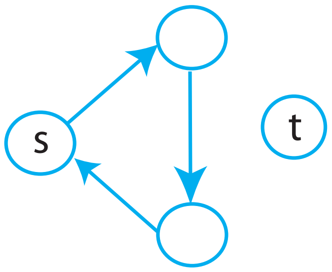


Figure 2: Shortest Paths

- works for directed acyclic graphs in $O(V+E)$
  effectively DFS/topological sort + Bellman-Ford round rolled into a single recursion

* Subproblem dependency should be acyclic

- more subproblems remove cyclic dependence:
  $\delta_k(s,v)$ = shortest $s \to v$ path using $\leq k$ edges
- recurrence:

$$\begin{aligned} \delta_k(s,v) &= \min\{\delta_{k-1}(s,u) + w(u,v) \big| (u,v) \in E\} \\ \delta_0(s,v) &= \infty \text{ for } s \neq v \text{ (base case)} \\ \delta_k(s,s) &= 0 \text{ for any } k \text{ (base case, if no negative cycles)} \end{aligned}$$

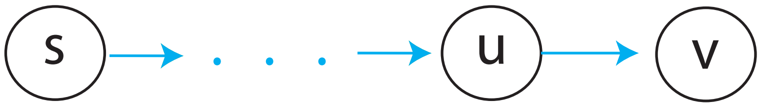
- <u>Goal</u>: $\delta(s,v) = \delta_{|V|-1}(s,v)$ (if no negative cycles)
- memoize
- time: $\underbrace{\#\text{ subproblems}}_{|V|\cdot|V|} \cdot \underbrace{\text{time/subproblem}}_{O(v)} = 0(V^3)$
- actually $\Theta(\text{indegree}(v))$ for $\delta_k(s,v)$
- $\implies$ time $= \Theta(V \sum_{v \in V} \text{indegree}(V)) = \Theta(VE)$

BELLMAN-FORD!

Guessing

How to design recurrence

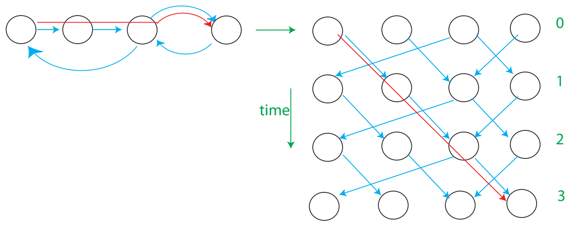
- want shortest $s \to v$ path



- what is the last edge in path? dunno
- <u>guess</u> it is (u, v)
- path is $\underbrace{\text{shortest } s \to u \text{ path}}_{\text{by optimal substructure}}$ + edge (u, v)
- cost is $\underbrace{\delta_{k-1}(s, u)}_{\text{another subproblem}} + w(u, v)$
- to find best guess, try all ($|V|$ choices) and use best
- * key: small (polynomial) # possible guesses per subproblem — typically this dominates time/subproblem

* DP $\approx$ recursion + memoization + guessing

DAG view



- like replicating graph to represent time
- converting shortest paths in graph $\to$ shortest paths in DAG

* DP $\approx$ shortest paths in some DAG