Final Exam - Basic Equations of Electrodynamics

Mathematical Identities

$v(t) = R_e\{\underline{V}e^{j\omega t}\}$ where $\underline{V} = |\underline{V}|e^{j\phi}$

$\nabla = \hat{x}\,\partial/\partial x + \hat{y}\,\partial/\partial y + \hat{z}\,\partial/\partial z$

$\overline{A}\cdot\overline{B} = A_x B_x + A_y B_y + A_z B_z$

$\nabla^2\phi = (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)\phi$

$\sin^2\theta + \cos^2\theta = 1$

$\nabla\cdot(\nabla\times\overline{A}) = 0$

$\nabla\times(\nabla\times\overline{A}) = \nabla(\nabla\cdot\overline{A}) - \nabla^2\overline{A}$

$\int_V (\nabla\cdot\overline{G})dv = \oint_S \overline{G}\cdot\hat{n}da$

$\int_S (\nabla\times\overline{G})\cdot\hat{n}da = \oint_C \overline{G}\cdot d\hat{s}$

$e^{j\omega t} = \cos\omega t + j\sin\omega t$

$\cos\alpha + \cos\beta = 2\cos[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]$

$\underline{H}(\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt$

$e^x = 1 + x + x^2/2! + x^3/3! + \dots$

$\sin\alpha = (e^{j\alpha} - e^{-j\alpha})/2j$

$\cos\alpha = (e^{j\alpha} + e^{-j\alpha})/2$

Radiation

$\overline{E} = -\nabla\phi - \partial\overline{A}/\partial t,\ \overline{B} = \nabla\times\overline{A}$

$\underline{\phi}(\overline{r}_p) = \int_{v_q} \{\rho(\overline{r}_q)e^{-jk|\overline{r}_{pq}|}/4\pi\varepsilon|\overline{r}_{pq}|\}dv_q$

$\underline{\overline{A}}(\overline{r}_p) = \int_{v_q} \{\mu_o \overline{J}(\overline{r}_q)e^{-jk|\overline{r}_{pq}|}/4\pi|\overline{r}_{pq}|\}dv_q$

$\nabla^2\underline{\phi} + \omega^2\mu_o\varepsilon_o\underline{\phi} = -\rho/\varepsilon_o$

$\nabla^2\underline{\overline{A}} + \omega^2\mu_o\varepsilon_o\underline{\overline{A}} = -\mu_o\underline{J}$

$\underline{\overline{E}}_{ff} = \hat{\theta}\eta_o(jk\underline{I}d/4\pi r)e^{-jkr}\sin\theta$

Electromagnetic Variables

$\overline{E}$ = electric field (V/m)

$\overline{H}$ = magnetic field (A/m)

$\overline{D}$ = electric displacement (C/m²)

$\overline{B}$ = magnetic flux density (T)

Tesla (T) = Weber/m² = 10^4 gauss

ρ = charge density (C/m³)

$\overline{J}$ = current density (A/m²)

σ = conductivity (Siemens/m)

$\overline{J}_s$ = surface current density (A/m)

ρ_s = surface charge density (C/m²)

Boundary Conditions

$\hat{n}\times(\overline{E}_1 - \overline{E}_2) = 0$

$\hat{n}\times(\overline{H}_1 - \overline{H}_2) = \overline{J}_s$

$\hat{n}\cdot(\overline{B}_1 - \overline{B}_2) = 0$

$\hat{n}\cdot(\overline{D}_1 - \overline{D}_2) = \rho_s$

$\underline{\overline{E}} = \underline{\overline{H}} = 0$ if $\sigma = \infty$

Maxwell's Equations, Force

$\nabla\times\overline{E} = -\partial\overline{B}/\partial t$

$\oint_C \overline{E}\cdot d\hat{s} = -\dfrac{d}{dt}\int_S \overline{B}\cdot\hat{n}da$

$\nabla\times\overline{H} = \overline{J} + \partial\overline{D}/\partial t$

$\oint_C \overline{H}\cdot d\overline{s} = \int_S \overline{J}\cdot\hat{n}da + \dfrac{d}{dt}\int_S \overline{D}\cdot\hat{n}da$

$\nabla\cdot\overline{D} = \rho \rightarrow \oint_S \overline{D}\cdot\hat{n}da = \int_V \rho dv$

$\nabla\cdot\overline{B} = 0 \rightarrow \oint_S \overline{B}\cdot\hat{n}da = 0$

$\nabla\cdot\overline{J} = -\partial\rho/\partial t$

$\overline{f} = q(\overline{E} + \overline{v}\times\mu_o\overline{H})$ [N]

Waves

$(\nabla^2 - \mu\varepsilon\,\partial^2/\partial t^2)\overline{E} = 0$

$(\nabla^2 + k^2)\underline{\overline{E}} = 0,\ \underline{\overline{E}} = \underline{\overline{E}}_o e^{-j\overline{k}\cdot\overline{r}}$

$k = \omega\sqrt{\mu\varepsilon} = \dfrac{\omega}{c} = \dfrac{2\pi}{\lambda} = \sqrt{k_x^2 + k_y^2 + k_z^2}$

$v_p = \omega/k,\quad v_g = \partial\omega/\partial k$

$E_x(z,t) = E_+(z-ct) + E_-(z+ct)$ [or $(\omega t-kz)$ or $(t-z/c)$]

$H_y(z,t) = (1/\eta_o)[E_+(z-ct) - E_-(z+ct)]$

$E_x(z,t) = R_e\{\underline{E}_x(z)e^{j\omega t}\}$

$<\overline{E}\times\overline{H}> = \dfrac{1}{2}Re\{\underline{\overline{E}}\times\underline{\overline{H}}^*\}$

$\oint_S (\overline{E}\times\overline{H})\cdot\hat{n}da = -\dfrac{d}{dt}\int_v (\dfrac{1}{2}\varepsilon|\overline{E}|^2 + \dfrac{1}{2}\mu|\overline{H}|^2)dv - \int_v \overline{E}\cdot\overline{J}dv$

Antennas

$P_r = P_t G_t G_r(\lambda/4\pi r)^2$

$G(\theta,\phi) = I(r,\theta,\phi)/(P_t/4\pi r^2)$

$R_r = P_t/<i^2(t)>$

$\underline{\overline{E}}_{ff} = [\hat{\theta}(j\eta_o d_{eff}/2\lambda r)\sin\theta]\sum \underline{I}_i e^{-jkr_i} = [\underline{\overline{\varepsilon}}(\theta,\phi)]\underline{F}(\theta,\phi)$

$A_e(\theta,\phi) = (\lambda^2/4\pi)G(\theta,\phi)$

$P_t = \iint I(\theta,\phi,r)r^2\sin\theta d\theta d\phi$

$P_r = I(\theta,\phi)A_e(\theta,\phi)$

$E_{bit} \geq \sim 4\times10^{-20}$ [J]

Constants

$\varepsilon_o = 8.85\times10^{-12}$ F/m

$\mu_o = 4\pi\times10^{-7}$ H/m

$c = 1/\sqrt{\varepsilon_o\mu_o} \cong 3\times10^8$ m/s

$h = 6.624\times10^{-34}$ Js

$e = 1.60\times10^{-19}$ [C]

$k_B = 1.38\times10^{-23}$ J/K

$\eta_o \cong 377\,\text{ohms} = \sqrt{\mu_o/\varepsilon_o}$

$m_e = 9.1066\times10^{-31}$ kg

Media

$\overline{D} = \varepsilon\overline{E} = \varepsilon_o\overline{E} + \overline{P}$

$\nabla\cdot\overline{D} = \rho_f$

$\nabla\cdot\varepsilon_o\overline{E} = \rho_f + \rho_p$

$\underline{\overline{D}} = \underline{\overline{\varepsilon E}},\quad \underline{\overline{J}} = \sigma\underline{\overline{E}}$

$\overline{B} = \mu\overline{H} = \mu_o(\overline{H} + \overline{M})$

$\varepsilon = \varepsilon_o(1 - \omega_p^2/\omega^2)$

$\omega_p = \sqrt{Ne^2/m\varepsilon_o}$

$\underline{\varepsilon}_{eff} = \varepsilon(1 - j\sigma/\omega\varepsilon)$

$\Delta = 2/(\sigma\eta)$

$\delta = \sqrt{2/\omega\mu\sigma}$

Planar Interfaces

$$\theta_i = \theta_r$$

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_i}{k_t} = \frac{\sqrt{\varepsilon_i\mu_i}}{\sqrt{\varepsilon_t\mu_t}} \triangleq \frac{n_i}{n_t}$$

$$\theta_c = \sin^{-1}(n_t/n_i)$$

$$\theta_B = \tan^{-1}\left(\frac{n_t}{n_i}\right)$$

$$1 + \underline{\Gamma} = \underline{T}$$

$$\underline{\Gamma}_{TE/TM} = (Z_n^{TE/TM} - 1)\big/(Z_n^{TE/TM} + 1)$$

$$Z_n^{TE} = \frac{\eta_t \cos\theta_i}{\eta_i \cos\theta_t}$$

$$Z_n^{TM} = \frac{\eta_t \cos\theta_t}{\eta_i \cos\theta_i}$$

$$k = k' + jk''$$

$$P_d \cong \left|\underline{J}_s\right|^2 / 2\sigma\delta \ [W/m^2]$$

Waveguides

$$\overline{\underline{E}}_{TE} = \hat{y}E_o \sin k_x x \cdot e^{-jk_z z}$$

$$\overline{\underline{E}}_{TE} = \hat{y}E_o \sin k_x x \cdot e^{-\alpha z}$$

$$\overline{\underline{E}}_{TE} = \hat{y}E_o e^{-\alpha|x|} e^{-jkz} (|x| > d)$$

$$\overline{\underline{E}}_{TE} = \hat{y}E_o \sin k_x x \cdot e^{-jkz} (|x| < d)$$

$$k_x^2 + k_z^2 = k_o^2 = \omega^2\mu\varepsilon$$

$$1/\lambda_g = 1/\lambda_z = \sqrt{1/\lambda_o^2 - 1/\lambda_x^2}$$

$$v_p = \frac{\omega}{k}, \quad v_g = \frac{d\omega}{dk}$$

Quasistatics

$$\overline{E} = -\nabla\Phi$$

$$\nabla^2\Phi = -\rho\big/\varepsilon_o$$

$$\underline{\Phi}(\overline{r}) = \int_{V'} \{\underline{\rho}(\overline{r}')/4\pi\varepsilon|\overline{r}' - \overline{r}|\}dv'$$

$$\mu_o\overline{H} = \nabla\times\overline{A}$$

$$\nabla^2\overline{A} = -\mu_o\overline{J}$$

$$\overline{\underline{A}}(r) = \int_{V'} \{\mu_o\overline{\underline{J}}(\overline{r}')/4\pi|\overline{r}' - \overline{r}|\}dv'$$

TEM Transients

$$\frac{dv(z,t)}{dz} = -L\frac{di(z,t)}{dt}$$

$$\frac{d^2v(z,t)}{dz^2} = LC\frac{d^2v(z,t)}{dt^2}$$

$$v(z,t) = f_+(t - \frac{z}{c}) + f_-(t + \frac{z}{c})$$

$$i(z,t) = Y_o\left(f_+(t - \frac{z}{c}) - f_-(t + \frac{z}{c})\right)$$

$$c = 1/\sqrt{LC} = 1/\sqrt{\mu\varepsilon}$$

$$Z_o = \sqrt{L/C}$$

$$\Gamma_L = \frac{R_L - Z_o}{R_L + Z_o} = \frac{R_{Ln} - 1}{R_{Ln} + 1}$$

$$v_{Th} = 2f_+(t), \quad R_{Th} = Z_o$$

Lasers

$$E = hf, \quad \frac{dn_2}{dt} = -An_2 - B(n_2 - n_1) \ [m^{-1}s^{-1}]$$

Acoustics: $c_s = \sqrt{\frac{\gamma P_o}{\rho_o}} \cong 330 \ [m/s]$, $\eta_s = \sqrt{\gamma P_o \rho_o}$, $\nabla p = -\rho_o\frac{\partial\overline{u}}{\partial t}$, $\nabla\bullet\overline{u} \cong -\frac{1}{\gamma P_o}\frac{\partial p}{\partial t}$

Circuit Elements

$$C = \frac{Q}{V}$$

$$L = \frac{\Lambda}{I}$$

$$i(t) = C\frac{dv(t)}{dt}$$

$$v(t) = L\frac{di(t)}{dt}$$

$$\Lambda = \int_A \overline{B}\cdot d\hat{a} \ \text{(per turn)}\cdot N$$

$$w_e(t) = \frac{1}{2}Cv^2(t)$$

$$w_m(t) = \frac{1}{2}Li^2(t)$$

$$\tau = RC, \quad \tau = \frac{L}{R}$$

TEM Sinusoidal Steady State

$$\underline{V}(z) = \underline{V}_+ e^{-jkz} + \underline{V}_- e^{+jkz}$$

$$\underline{I}(z) = Y_o(\underline{V}_+ e^{-jkz} - \underline{V}_- e^{+jkz})$$

$$k = 2\pi/\lambda = \omega/c = \omega\sqrt{\mu\varepsilon}$$

$$\underline{Z}(z) = \underline{V}(z)/\underline{I}(z) = Z_o\cdot\underline{Z}_n(z)$$

$$\underline{\Gamma}(z) = (V_-/V_+)e^{2jkz} = (\underline{Z}_n(z)-1)/(\underline{Z}_n(z)+1)$$

$$\underline{Z}_n(z) = [1 + \underline{\Gamma}(z)]/[1 - \underline{\Gamma}(z)] = R_n + jX_n$$

$$\underline{Z}(z) = Z_o\cdot(Z_L - jZ_o\tan kz)/(Z_o - jZ_L\tan kz)$$

$$VSWR = \left|\underline{V}_{max}\right|/\left|\underline{V}_{min}\right| = \frac{1 + |\underline{\Gamma}|}{1 - |\underline{\Gamma}|} = R_{n\,max}$$

Electromagnetic Forces

$$\overline{f} = q(\overline{E} + \overline{v}\times\mu_o\overline{H}) \ [N]$$

$$\overline{F} = \overline{I}\times\mu_o\overline{H} \ [N/m]$$

$$\overline{E}_e = -v\times\mu_o\overline{H} \ \text{(inside conductor)}$$

$$vi = \frac{dw}{dt} + f\frac{dz}{dt}$$

$$f_x = -\frac{dw_e}{dx}\Big|_{Q=const.}$$

$$f_x = -\frac{dw_m}{dx}\Big|_{\Lambda=const.}$$

$$\overline{T} = \overline{r}\times\overline{f} \quad P_e = \mu H^2/2, \ \varepsilon E^2/2 \ [N/m^2]$$

$$T_\theta = -\frac{dw}{d\theta}\Big|_{Q \ or \ \Lambda=const}$$

RLC Resonators

$$Z_{series} = R + j\omega L + 1/j\omega C$$

$$Y_{parallel} = G + j\omega C + 1/j\omega L$$

$$\omega_o = 1/\sqrt{LC}, \quad Q = \frac{\omega_o w_T}{P_{diss}} = \frac{\omega_o}{\Delta\omega}$$

EM Resonators

$$\text{At } \omega_o, \ \langle w_e \rangle = \langle w_m \rangle$$

$$\langle w_e \rangle = \int_V (\varepsilon|\underline{E}|^2/4)dv$$

$$\langle w_m \rangle = \int_V (\mu|\underline{H}|^2/4)dv$$

$$f_{mnp} = \frac{c}{2}\sqrt{(m/a)^2 + (n/b)^2 + (p/d)^2}$$

$$Q = \frac{\omega_o w_T}{P_{diss}} = \frac{\omega}{2\alpha} = \frac{\omega}{\Delta\omega}, \quad \frac{1}{Q_L} = \frac{1}{Q_E} + \frac{1}{Q_I}$$

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