Problem Set 7 Solutions

Problem 7.1

(a)

(i) For an air filled line we know that the propagation velocity is $c_\circ$. Looking at the figure we see the distance between adjacent voltage minima is A, so the wavelength is $\lambda = 2A$. The frequency of the wave is given by the ratio of the propagation velocity to the wavelength:

$$f = \frac{v_p}{\lambda} = \frac{3\times10^8}{2A} = \frac{1.5}{A} \times 10^8 \ [Hz]$$

(ii) The voltage standing wave ratio is the ratio of the maximum voltage on the line to the minimum voltage on the line:

$$VSWR = \frac{|\underline{V}(z)|_{max}}{|\underline{V}(z)|_{min}}$$

(iii) At a voltage minima (at the load in this case) we know that $\underline{\Gamma} = -|\underline{\Gamma}| = \frac{\underline{Z}_n - 1}{\underline{Z}_n + 1}$. So we can write:

$$\underline{Z}(z=0)_N = \frac{R_L}{Z_\circ} = \frac{1+\underline{\Gamma}}{1-\underline{\Gamma}} = \frac{1-|\underline{\Gamma}|}{1+|\underline{\Gamma}|} = \frac{1}{VSWR}$$

We can solve this for $Z_\circ$

$$Z_\circ = \frac{R_L}{\underline{Z}(z=0)_n} = R_L \times VSWR = R_L \frac{|\underline{V}(z)|_{max}}{|\underline{V}(z)|_{min}}$$

(iv) We know that the velocity of propagation on the line is $V_p = \frac{1}{\sqrt{LC}}$, and that the characteristic impedance of the line is $Z_\circ = \sqrt{\frac{L}{C}}$, so the Capacitance per unit length of the line is

$$C = \frac{1}{V_p Z_\circ} = \frac{1}{(3\times10^8)R_L VSWR} \ [F/m]$$

(v) Using the same relations as in part (iv) we know that the inductance per unit length will be

$$L = \frac{Z_\circ}{V_p} = \frac{R_L VSWR}{3\times10^8}$$

(vi) We know that the magnitude of the reflection coefficient is given by

$$|\underline{\Gamma}| = \frac{VSWR-1}{VSWR+1}$$

and that at $z = 0$ the reflection coefficient is $\underline{\Gamma} = -|\underline{\Gamma}$

So the reflection coefficient is

$$\underline{\Gamma}(z) = -\frac{VSWR-1}{VSWR+1} e^{j2\pi z/A}$$

(vii) The fraction of the power reflected by the load is the square of the magnitude of the reflection coefficient

$$\frac{P_-}{P_+} = \left(\frac{VSWR-1}{VSWR+1}\right)^2$$

(b) In this case, we can calculate VSWR and $|\underline{\Gamma}|$.

$$VSWR = \frac{2}{1} = 2, \ |\underline{\Gamma}| = \frac{VSWR-1}{VSWR+1} = \frac{2-1}{2+1} = \frac{1}{3}$$

Again, at a voltage minimum the reflection coefficient is $\underline{\Gamma}|_{Vmin} = -|\Gamma| = -\frac{1}{3}$. So at the voltage minimum, the normalized impedance is

$$\underline{Z}|_{Vmin} = \frac{1+\underline{\Gamma}}{1-\underline{\Gamma}} = \frac{1-1/3}{1+1/3} = \frac{1}{2}$$

From this point we can either solve the problem using a Smith Chart or by using equation 7.2.24.

To solve using a Smith Chart first find the point corresponding to $Z_N = 0.5$. Next draw the $|\Gamma| = const$ circle. Finally, rotate from the $Z_N = 0.5$ point toward the load by $\frac{\lambda}{8}$. The new point is the normalized load impedance at the load. Based on the attached Smith Chart, $Z_L = Z_\circ(0.8 - 0.6j)$.

To solve using equation 7.2.24, first observe that when $z = -\frac{\lambda}{8}$, $tan(kz) = tan(\frac{2\pi}{\lambda}\frac{-\lambda}{8}) = tan(-\pi/4) = -1$

$\underline{Z}(\frac{-\lambda}{8})_N = \frac{1}{2} = \frac{Z_L + jZ_\circ}{Z_\circ + jZ_L}$

$0.5Z_\circ + 0.5jZ_L = Z_L + jZ_\circ$

$Z_L(1 - 0.5j) = Z_\circ(0.5 - j)$

$Z_L = Z_\circ \frac{0.5-j}{1-0.5j} = Z_\circ \frac{1-2j}{2-j} \times \frac{2+j}{2+j} = Z_\circ \frac{4-3j}{5} = Z_\circ(0.8 - 0.6j)$

Which is the same as the answer from the Smith Chart.

(c) For shunt matching it is easier to work with admittances. The admittance of a capacitor is $Y_C = j\omega C > 0$, so we need to find the first point on the line where the input admittance as a real part of 1 (matched) and negative imaginary part (so that the admittance of the capacitor can cancel it out). Again, it's possible to solve this problem using a equation 7.2.24 or the Smith Chart.

To solve using the Smith Chart, first locate the load impedance. Next draw the circle of constant $|\underline{\Gamma}|$, and find the point $\lambda/2$ from the load. This is the admittance of the load. Follow the constant $|\underline{\Gamma}|$ circle to the point it crosses the $Re(\underline{Y}) = 1$ circle, with a negative imaginary value. This is the point at which we can match the load with a shunt capacitance.

From the Smith Chart, we can match at $z = -0.2225\lambda$ with a capacitance $C = 0.7/(\omega Z_\circ)$.

We can also solve this problem using equation 7.2.24. First we solve for the normalized admittance:

$\underline{Y}_N = \frac{1}{\underline{Z}_N} = \frac{Z_\circ - j\underline{Z}_L tan(kz)}{\underline{Z}_L - jZ_\circ tan(kz)} = \frac{Z_\circ - j\underline{Z}_L tan(kz)}{\underline{Z}_L - jZ_\circ tan(kz)} \frac{\underline{Z}_L^* + jZ_\circ tan(kz)}{\underline{Z}_L^* + jZ_\circ tan(kz)} = \frac{Z_\circ \underline{Z}_L^* + jZ_\circ^2 tan(kz) - j\underline{Z}_L \underline{Z}_L^* tan(kz) + \underline{Z}_L Z_\circ tan^2(kz)}{|\underline{Z}_L|^2 - 2Z_{LI} Z_\circ tan(kz) + Z_\circ^2 tan^2(kz)}$

Where Z_{LI} is the imaginary part of the load impedance. To match the load, we need to find a place where the real part of the complex admittance is 1.

$Re(\underline{Y}_N) = 1 = \frac{Z_\circ Z_{LR} + Z_{LR} Z_\circ tan^2(kz)}{|\underline{Z}_L|^2 - 2Imag(Z_L) Z_\circ tan(kz) + Z_\circ^2 tan^2(kz)}$

We can plug in the known load impedance, and divide top and bottom by $Z_\circ^2$ to get:

$1 = \frac{0.8 + 0.8tan^2(kz)}{1 + 1.2tan(kz) + tan^2(kz)}$

$tan^2(kz) + 1.2tan(kz) + 1 = 0.8 + 0.8tan^2(kz)$

$0.2tan^2(kz) + 1.2tan(kz) + 0.2 = 0$

$tan^2(kz) + 6tan(kz) + 1 = 0$

We can solve for $tan(kz)$ using the quadratic formula

$tan(kz) = \frac{-6 \pm \sqrt{6^2 - 4*1*1}}{2*1} = -3 \pm 2\sqrt{2}$

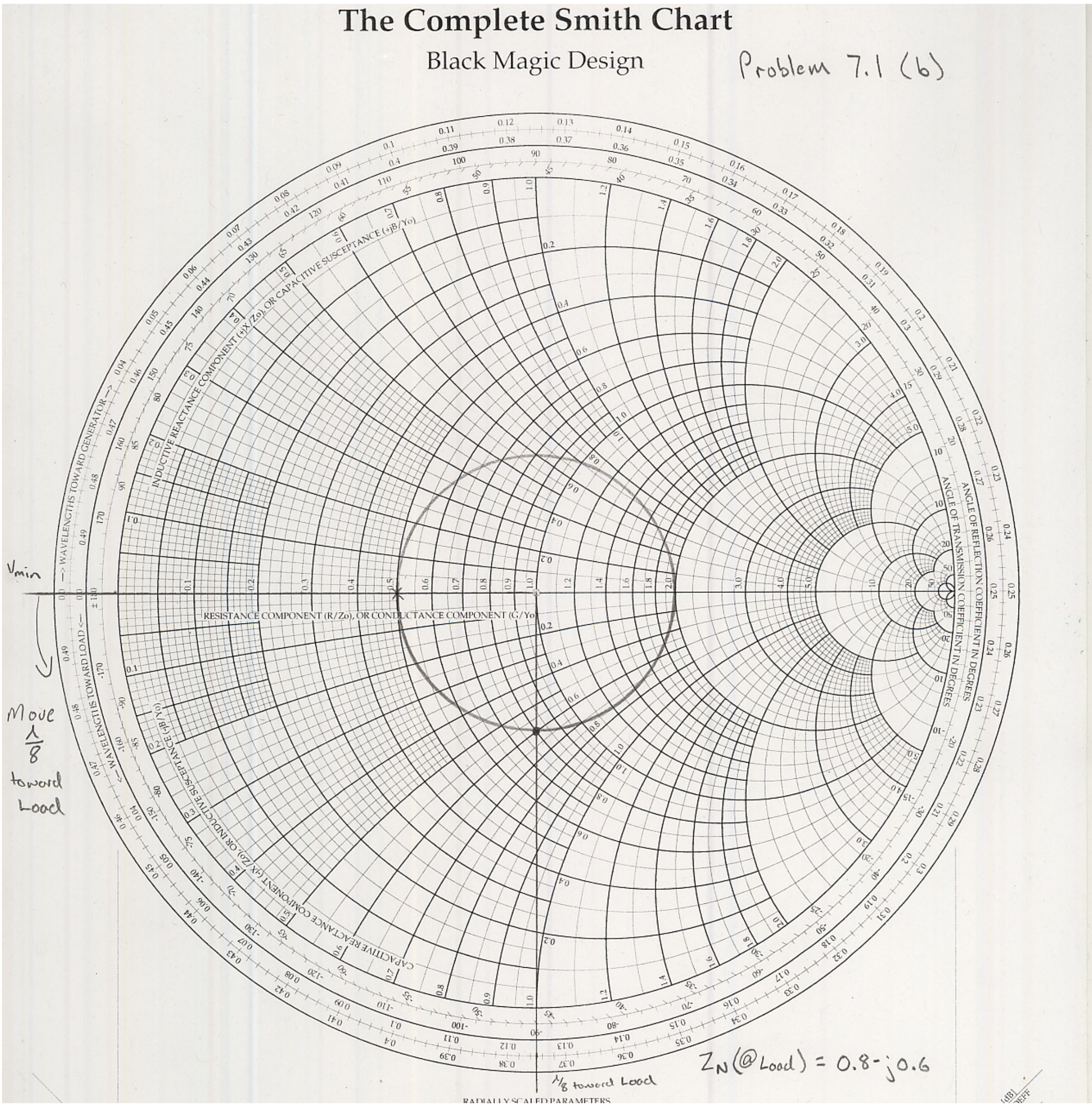
$z = \frac{\lambda}{2\pi} arctan(-3 \pm 2\sqrt{2})$

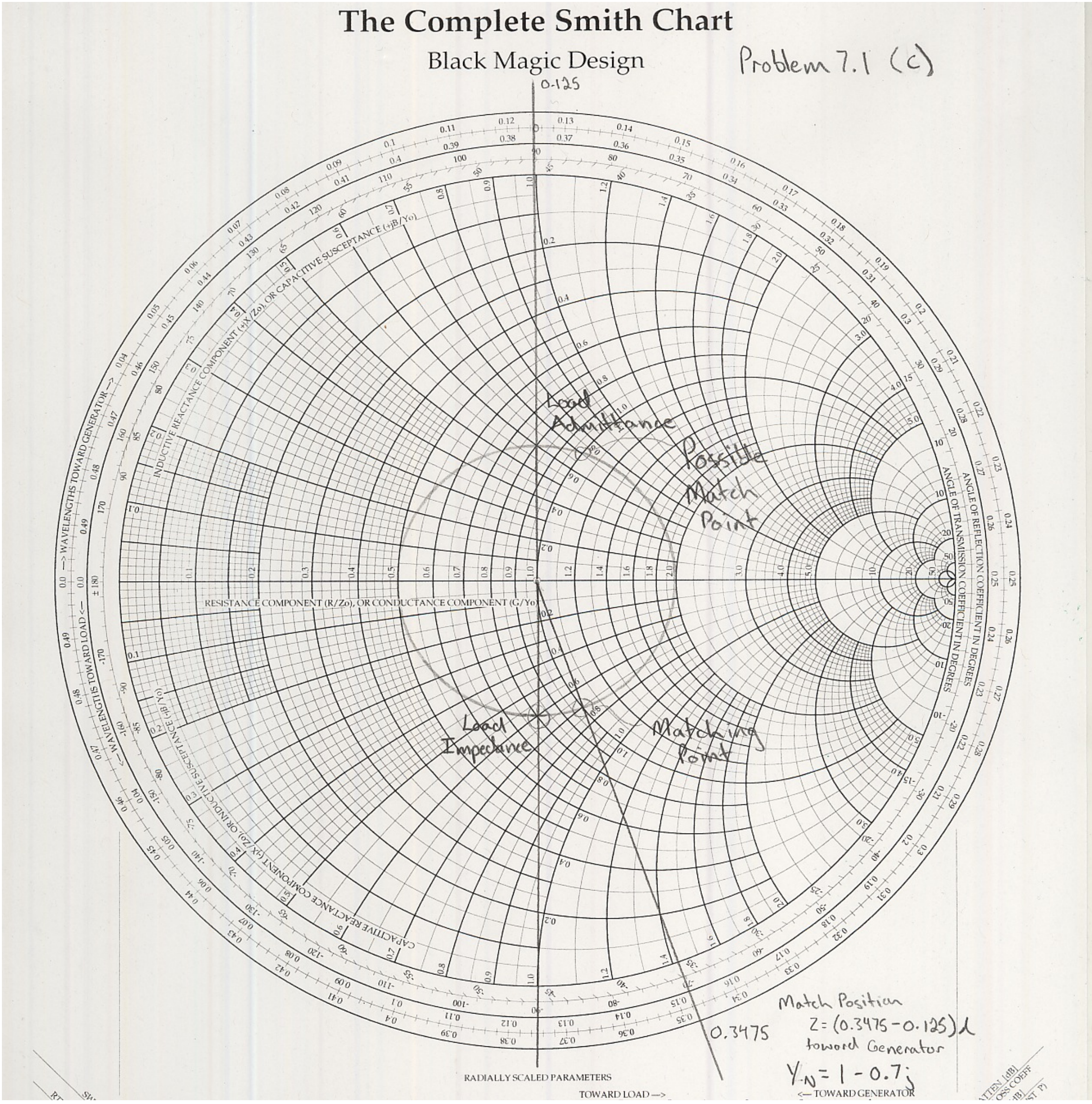
Giving two possible solutions $z = -0.02704\lambda$, and $z = -0.22296\lambda$. Adding some multiple of 0.5λ to either of these solutions will also satisify the $Re(\underline{Y}) = 1$ condition since $tan(kz)$ is periodic, so we actually have an infinite number of possible match points. For simplicity we'll take the first point that meets our requirements.

By plugging these distances into the impedance equation, we see that $z = -0.22296\lambda$ is the solution we want. At this point the input admittance is:

$\underline{Y}_n = \frac{1-j(0.8-j0.6)tan(-2\pi 0.22296)}{(0.8-j0.6)-jtan(-2\pi 0.22296)} \approx 1 - j0.7071$

So we can match the load by placing a shunt capacitance of $C = 0.7071/(\omega Z_\circ)$ at position $z = -0.22296\lambda$




The Complete Smith Chart
Black Magic Design
Problem 7.1 (c)
0.125
Load Admittance
Possible Match Point
Load Impedance
Matching Point
0.3475
Match Position
Z= (0.3475 - 0.125)λ
toward Generator
Y_N = 1 - 0.7j
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo) OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES
—> WAVELENGTHS TOWARD GENERATOR —>
<— WAVELENGTHS TOWARD LOAD <—
RADIALLY SCALED PARAMETERS
TOWARD LOAD —>
<— TOWARD GENERATOR

Problem 7.2

(a) From the VSWR we can determine the magnitude of the reflection coefficient. The point where the bulb is brightest will correspond to a current maximum ($\underline{\Gamma}(z) = -|\underline{\Gamma}|$).

$|\underline{\Gamma}| = \frac{VSWR-1}{VSWR+1} = \frac{3-1}{3+1} = \frac{2}{4} = \frac{1}{2}$

$\underline{\Gamma}(-\lambda/4) = -\frac{1}{2} = \underline{\Gamma}_L e^{-j2k\lambda/4} = \underline{\Gamma}_L e^{-j2\frac{2\pi}{\lambda}\frac{\lambda}{4}}$

$\underline{\Gamma}_L = -\frac{1}{2}e^{j\frac{4\pi}{\lambda}\frac{\lambda}{4}} = -\frac{1}{2}e^{j\pi} = \frac{1}{2}$

$\underline{Z}_{LN} = \frac{1+\underline{\Gamma}_L}{1-\underline{\Gamma}_L} = \frac{1.5}{0.5} = 3$

$\underline{Z}_L = \underline{Z}_{LN} Z_\circ = 3 \times 300 = 900\ [\Omega]$

(b) It's possible to match the load using a short-circuited stub in series or in parallel with the antenna feed line. The position and length of the stub will likely be different in each case, so we need to test both methods and pick the one that results in the shortest stub length.

We could work the problem exactly for each matching statagie, or we can use the Smith Chart to get the approximate solutions and then do the exact solution for the case predicted to be shorter. Since I like using the Smith Chart that's how we'll solve the problem. The Smith Chart solutions are attached at the end of this problem solution. The method for each solution is described below.

Smith Chart Series Match

To find a match using the stub in series we work with impedances.

(1) First locate the normalized load point on the Smith Chart.

(2) Next draw the circle of constant $|\underline{\Gamma}|$. There are two possible match positions. These positions occur when the circle of constant $|\underline{\Gamma}|$ crosses the $Re\{\underline{Z}_n\} = 1$ circle. The imaginary component of the short-circuited stub impedance goes positive (from zero) before going negative, so we want the match point where the load has a negative imaginary component of impedance.

(3) From the match point we can read off the value of the imaginary component of impedance ($-j1.16Z_\circ$), and get the position ($(0.333-0.25)\lambda = \lambda/12$ towards generator). Next we find the length of the stub needed to match the load at this point.

(4) First find the stub impedance point ($Z_n = 0$, position $= 0\ \lambda$ towards generator). Next find the point corresponding to an impedance of $j1.16Z_\circ$ (position $= 0.136\ \lambda$ towards generator). The difference in positions is the length of the stub needed to match the load ($0.136\lambda = 0.544\ [m]$).

Smith Chart Parallel Match

To find a match using the stub in parallel it is easier to work with admittances.

(1) First locate the normalized load point on the Smith Chart.

(2) Next draw the circle of constant $|\underline{\Gamma}|$. Rotate about this circle by $\lambda/4$ to find the normalized load admittance.

(3) Identify the two possible match positions. These positions occur when the circle of constant $|\underline{\Gamma}|$ crosses the $Re\{\underline{Y}_n\} = 1$ circle. The imaginary component of the short-circuited stub admittance goes negative (from infinity) before going positive, so we want the match point where the load has a positive imaginary component of admittance.

(4) From the match point we can read off the value of the imaginary component of admittance ($+j1.155Z_\circ$), and get the position (0.1675λ towards generator). Next we find the length of the stub needed to match the

load at this point.

(5) First find the stub impedance point ($Z_n = 0$, position $= 0\ \lambda$ towards generator). Now rotate by $\lambda/4$ to get the stub admittance (infinity, position $\lambda/4$). Next find the point corresponding to an admittance of $j - 1.155Z_\circ$ (position $= 0.363\ \lambda$ towards generator). The difference in positions is the length of the stub needed to match the load ($0.113\lambda = 0.4544\ [m]$).

From the Smith Chart results, we know that we can match the load using the shortest short-circuited stub when we place the stub in parallel with the feed line.

Exact Solution to the Parallel Match

$\underline{Y}_n(z) = \frac{Z_\circ - jZ_L tan(kz)}{Z_L - jZ_\circ tan(kz)}$

$\underline{Y}_n(z) = \frac{Z_\circ - jZ_L tan(kz)}{Z_L - jZ_\circ tan(kz)} * \frac{Z_L + jZ_\circ tan(kz)}{Z_L + jZ_\circ tan(kz)} = \frac{1}{Z_L^2 + Z_\circ^2 tan^2(kz)}\ Z_\circ Z_L + jZ_\circ^2 tan(kz) - jZ_L^2 tan(kz) + Z_L Z_\circ tan^2(kz)$

$Re\{\underline{Y}_n(z)\} = 1 = \frac{1}{Z_L^2 + Z_\circ^2 tan^2(kz)}\ Z_\circ Z_L + Z_L Z_\circ tan^2(kz)$

$Z_L^2 + Z_\circ^2 tan^2(kz) = Z_\circ Z_L + Z_L Z_\circ tan^2(kz)$

$Z_L^2 - Z_\circ Z_L = tan^2(kz)(Z_L Z_\circ - Z_\circ^2)$

$tan(kz) = \pm\sqrt{\frac{Z_L^2 - Z_\circ Z_L}{Z_L Z_\circ - Z_\circ^2}} = \pm\sqrt{\frac{900^2 - 300*900}{900*300 - 300^2}} = \pm\sqrt{3}$

$kz = \pm\frac{\pi}{3}$,

$z = -\frac{\lambda}{6}$ (this is the sign that will work for us).

Plugging this back into the expression for impedance we get

$\underline{Y}(-\lambda/6) = Y_\circ(1 + j\sqrt{3}\frac{2}{3})$

So our stub has to have an admittance of $-j\sqrt{3}\frac{2}{3}Y_\circ$.

$\underline{Y}_S = \frac{jY_\circ}{tan(kz)} = -j\sqrt{3}\frac{2}{3}Y_\circ$

$tan(kz) = -\frac{\sqrt{3}}{2}$

$\frac{2\pi}{\lambda}z = tan^{-1}(-\frac{\sqrt{3}}{2})$

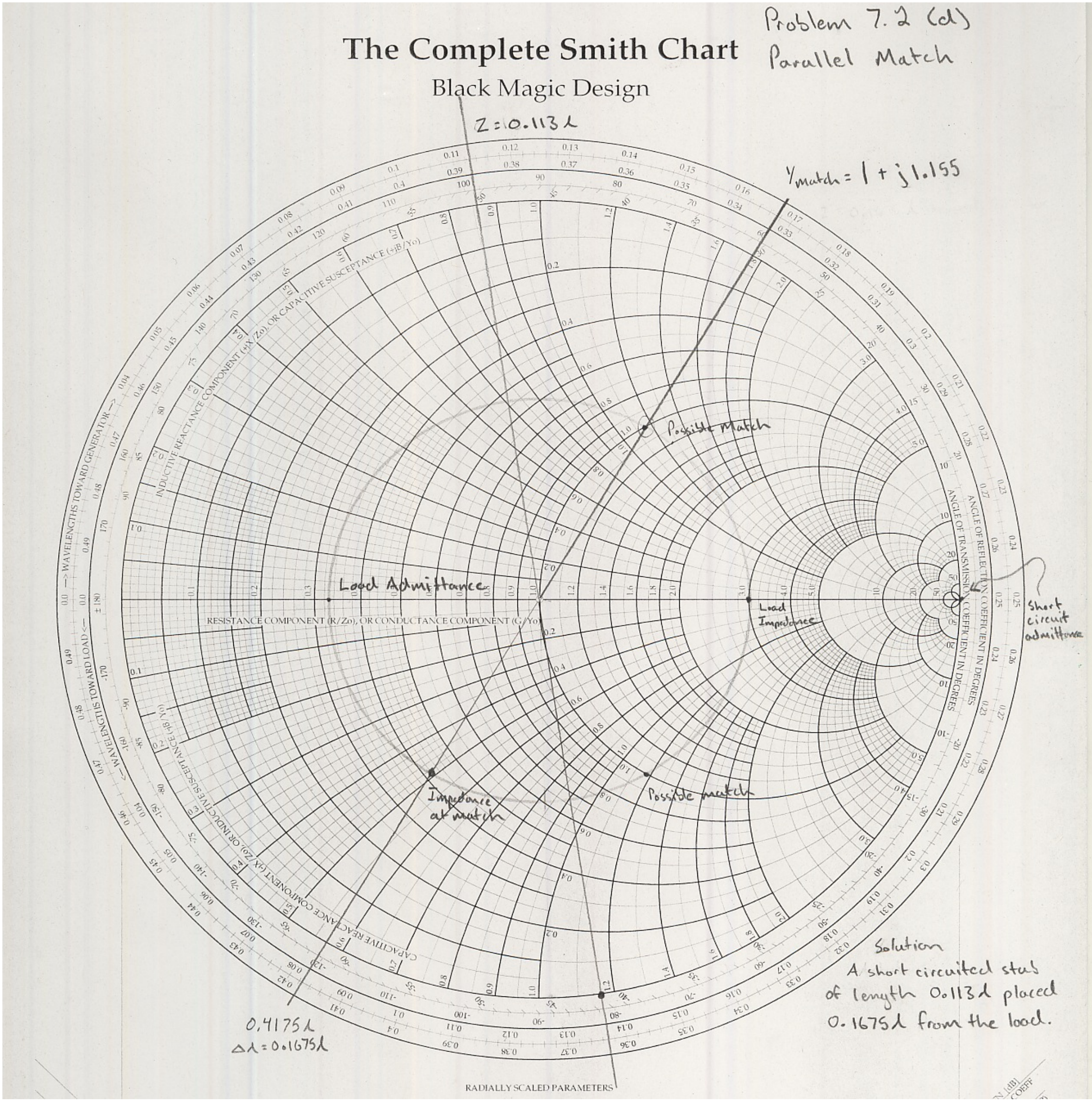
$z = \frac{\lambda}{2\pi}tan^{-1}(-\frac{\sqrt{3}}{2}) = -0.1136\lambda = -0.4544\ [m]$

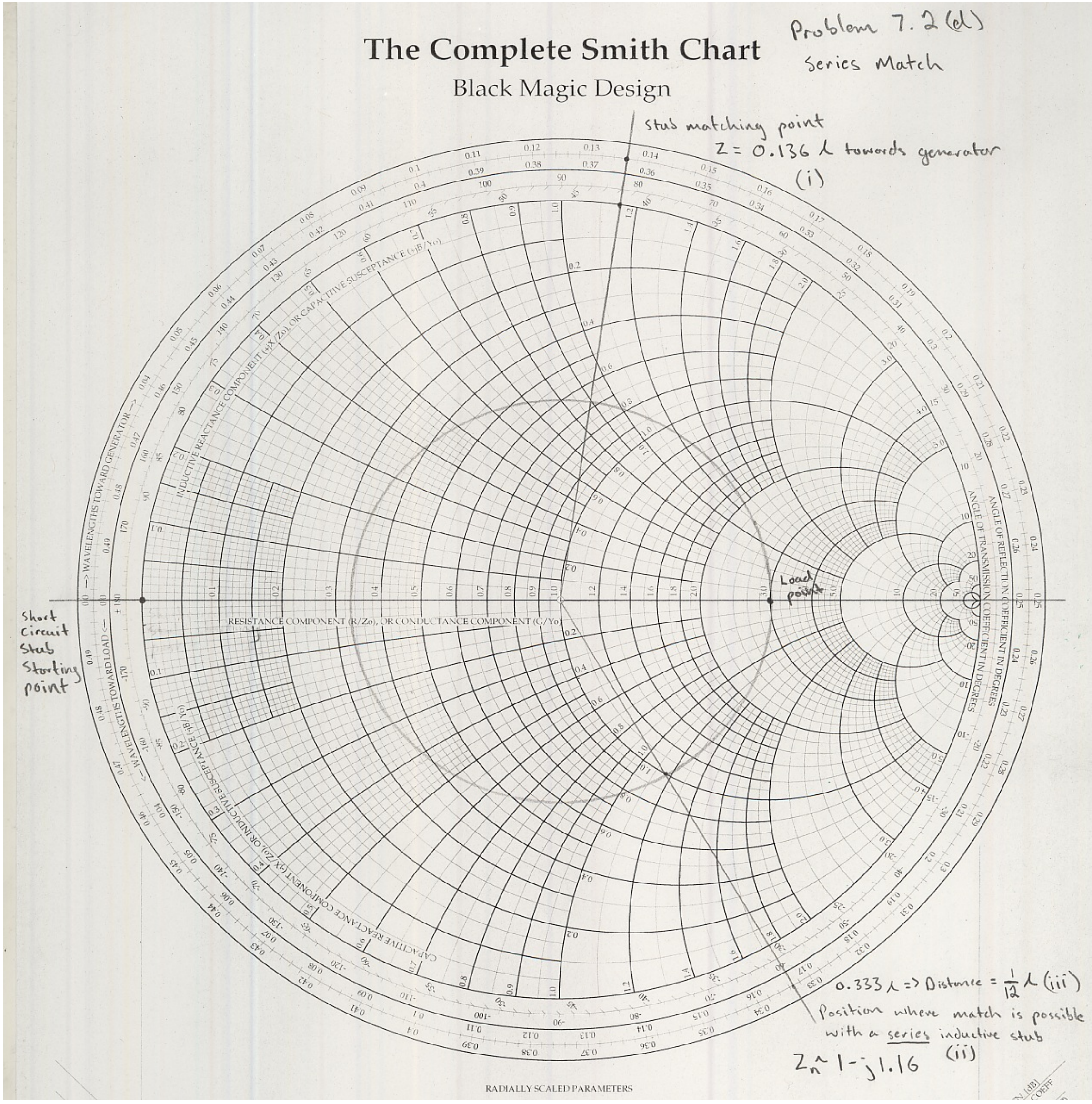
So,

(i) The stub should be 0.4544 meters long

(ii) The stub should be in parallel with the feed line

(iii) The stub should be placed $\frac{\lambda}{6} = \frac{2}{3}$ meters from the antenna port


The Complete Smith Chart
Black Magic Design
Problem 7.2 (d)
Parallel Match
Z=0.113λ
Ymatch = 1 + j1.155
Possible Match
Load Admittance
Load Impedance
Short circuit admittance
Impedance at match
Possible match
Solution
A short circuited stub of length 0.113λ placed 0.1675λ from the load.
0.4175λ
Δλ=0.1675λ
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
RADIALLY SCALED PARAMETERS


The Complete Smith Chart
Black Magic Design
Problem 7.2 (d)
Series Match
Stub matching point
Z = 0.136 λ towards generator
(i)
Load point
Short circuit stub starting point
0.333 λ => Distance = 1/12 λ (iii)
Position where match is possible with a series inductive stub
(ii)
Zn ≈ 1 - j1.16
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
RADIALLY SCALED PARAMETERS

Problem Set 7 Draft Solution Problem 7.3

Problem 7.3

(a) For a half-wave line, the input impedance is just the load impedance.

(b) For a $\frac{3}{4}$-wave line, the input impedance is the inverse of the load impedance.

$\underline{Z}_{in} = \frac{1}{Z_\circ - jZ_\circ} = \frac{1}{Z_\circ}\frac{1}{1-j} * \frac{1+j}{1+j} = \frac{1}{Z_\circ}\frac{1+j}{2} = \frac{0.5}{Z_\circ} + j\frac{0.5}{Z_\circ}$

At the input, the impedance looks like an inductor in series with a resistor.

(c) In this case the problem is a little more complicated. The Thevenin equivalent resistance will be $Z_\circ$. To find the open circuit voltage, we need to think about the standing wave pattern that results in the steady state.

At the input (from the sources point of view) of the transmission line, the line's equivalent impedance is zero (the $\lambda/4$ transform of an open circuit). So the current has to be $I_T = \frac{\underline{V}_S}{\underline{Z}_\circ}$. This gives us two equations:

$V_T = 0 = V_+ + V_- = (I_+ + I_-)Z_\circ$

$I_T = \frac{\underline{V}_S}{\underline{Z}_\circ} = I_+ - I_-$

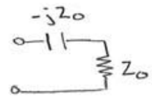
The first equation tells us that $I_- = -I_+$, and when we put this into the second equation we get $I_+ = \frac{\underline{V}_S}{2Z_\circ}$. Using this we can get $V_+ = \frac{\underline{V}_S}{2}$. At the open circuit the voltage will be twice the forward going voltage (since the reflection coefficient is 1 at an open circuit). Finally, we know that the open circuit voltage will lag behind the source voltage by a quarter wave ($\phi = kz = -\frac{2\pi}{\lambda}\frac{\lambda}{4} = -\frac{\pi}{2}$).

So the Thevenin voltage will be $V_{Thevenin} = \underline{V}_S\ e^{-j\frac{\pi}{2}}$.

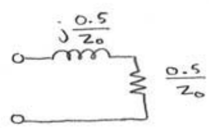
(d) No solution, skip this problem

Problem 7.3 diagrams

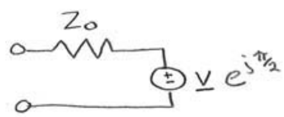
(a)



(b)



(c)



(d)

Problem Set 7 Draft Solution Problem 7.4

Problem 7.4

(a) Assume we want to match at normal incidence, and at the center of the wavelength band given ($0.5\mu m$). Further, assume that we can neglect reflections from the back of the solar cell. We can approximate this problem as a quarter wave line matching problem. The impedance of one line will be $\underline{Z}_\circ = \eta_\circ$ and the impedance of the second line will be $\underline{Z}_L = \eta_L$.

We know from the notes that the impedance of the line used to match the two materials should be $\underline{Z}_a = (Z_\circ Z_L)^{0.5}$, so we have the following relation

$\eta_a = \sqrt{\frac{\mu_\circ}{\varepsilon_a}} = (\sqrt{\frac{\mu_\circ}{\varepsilon_\circ}}\sqrt{\frac{\mu_\circ}{\varepsilon_L}})^{0.5} = (\frac{\mu_\circ}{\sqrt{\varepsilon_\circ \varepsilon_L}})^{0.5}$

So we need to choose

$\varepsilon_a = \sqrt{\varepsilon_\circ \varepsilon_L} = 2\varepsilon_\circ$

For this value of ε, a free-space wavelength of $0.5\mu m$ has wavelength $\lambda = \lambda_\circ/\sqrt{2} = \frac{0.5}{\sqrt{2}}$ $[\mu m]$. The coating thickness will be one quarter of this value.

$d = \frac{1}{8\sqrt{2}}$ $[\mu m] \approx 88$ $[nm]$

(b) At one micron, the layer will be $\frac{\lambda}{8}$ thick, so we know the apparent impedance at the surface of the coating will be

$Z_{in} = Z_a \frac{Z_L + jZ_a}{Z_a + jZ_L} = \eta_\circ \varepsilon_a^{-0.5} \frac{\eta_\circ \varepsilon_a^{-0.5} + j\eta_\circ \varepsilon_L^{-0.5}}{\eta_\circ \varepsilon_L^{-0.5} + j\eta_\circ \varepsilon_a^{-0.5}} = \eta_\circ \varepsilon_a^{-0.5} \frac{\varepsilon_a^{0.5} + j\varepsilon_L^{0.5}}{\varepsilon_L^{0.5} + j\varepsilon_a^{0.5}}$

$\frac{Z_{in}}{Z_\circ} = \frac{1}{\sqrt{2}} \frac{\sqrt{2} + j2}{2 + j\sqrt{2}}$

$\underline{\Gamma} = \frac{\frac{Z_{in}}{Z_\circ} - 1}{\frac{Z_{in}}{Z_\circ} + 1}$

The fraction of the power reflected is $|\underline{\Gamma}|^2$, which is

$|\underline{\Gamma}|^2 = 0.059$, at one micron.

Problem Set 7 Draft Solution Problem 7.5

Problem 7.5

(a) For a series RLC resonance, we have the following relations

$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$ which tells us $\sqrt{\frac{L}{C}} = RQ = 2000.$

$\omega = \frac{1}{\sqrt{LC}} = 2\pi \times 10^6$ which tells us $\sqrt{LC} = (2\pi \times 10^6)^{-1}.$

$L = \sqrt{LC}\sqrt{\frac{L}{C}} = (2\pi \times 10^6)^{-1} \times 2000 = \frac{1}{\pi} \times 10^{-3}$

$C = \sqrt{LC}\sqrt{\frac{C}{L}} = (2\pi \times 10^6)^{-1} \times \frac{1}{2000} = \frac{1}{4\pi} \times 10^{-9}$

(b) For a parallel RLC resonance, we have the following relations

$Q = R\sqrt{\frac{C}{L}}$ which tells us $\sqrt{\frac{C}{L}} = Q/R = 0.2$

$\omega = \frac{1}{\sqrt{LC}} = 2\pi \times 10^6$ which tells us $\sqrt{LC} = (2\pi \times 10^6)^{-1}.$

$L = \sqrt{LC}\sqrt{\frac{L}{C}} = (2\pi \times 10^6)^{-1} \times \frac{1}{0.2} = \frac{1}{4\pi} \times 10^{-5}$

$C = \sqrt{LC}\sqrt{\frac{C}{L}} = (2\pi \times 10^6)^{-1} \times 0.2 = \frac{1}{\pi} \times 10^{-7}$

(c) From the problem statement we know we are looking for a parallel resonator (short circuit far from resonance), with the input to the amplifier serving as the load resistor. We also know that $Q_{Loaded} = \frac{2\pi\times 1\times 10^6}{2\pi\times 50\times 10^6} = 0.02.$

$Q_{Loaded} = 0.02 = \omega\frac{CR_{th}R}{R_{th}+R}$, which tells us that

$C = \frac{Q}{\omega}\frac{R_{th}+R}{R_{th}R} = \frac{1}{2\pi\times 50\times 10^6}\frac{200}{100^2} = \frac{4}{\pi} \times 10^{-10}$

$\omega = \frac{1}{\sqrt{LC}}$ which tells us $L = \frac{1}{\omega^2 C}.$

$L = \frac{1}{(2\pi\times 10^6)^2\frac{4}{\pi}\times 10^{-10}} = \frac{1}{16\pi} \times 10^{-2}$