Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

Recitation 11 Solutions
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1. Let P be the random variable for the value drawn according to the uniform distribution in interval $[0, 1]$ and let X be the number of successes in k trials. Given $P = p$, X is a binomial random variable:

$$p_{X|P}(x|p) = \begin{cases} \binom{k}{x} p^x (1-p)^{k-x} & x = 0, 1, ..., k \\ 0 & \text{otherwise.} \end{cases}$$

From the properties of a binomial r.v. we know that $E[X|P = p] = kp$, and $Var(X|P = p) = kp(1 - p)$. So,

$$\begin{aligned} E[X^2|P = p] - (E[X|P = p])^2 &= kp(1-p) \\ E[X^2|P = p] - (kp)^2 &= kp(1-p) \\ E[X^2|P = p] &= kp(1-p) + (kp)^2. \end{aligned}$$

Let's find $E[X]$ using the iterated expectation law:

$$\begin{aligned} E[X] &= E[E[X|P]] \\ &= E[kP] \\ &= kE[P] \\ &= \frac{k}{2} \end{aligned}$$

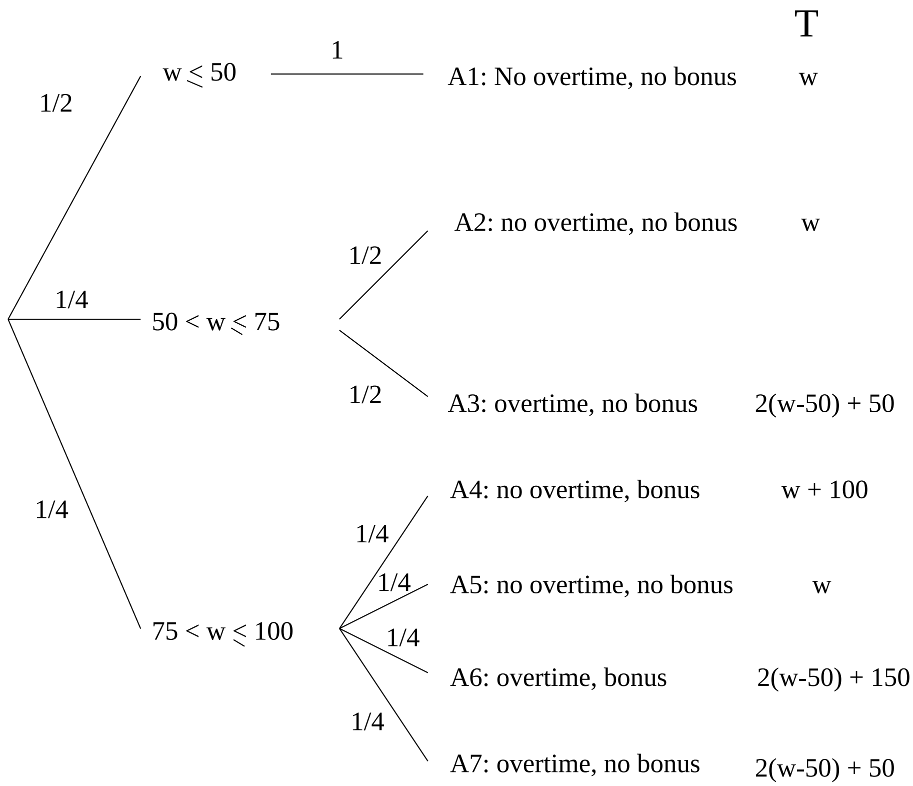
Now let's find $\text{var}(X)$ using the law of total variance:

$$\begin{aligned} var(X) &= E[var(X|P)] + var(E[X|P]) \\ &= E[kP(1-P)] + var(kP) \\ &= k(E[P] - E[P^2]) + k^2 var(P) \\ &= k[\frac{1}{2} - (\frac{1}{12} + 14)] + k^2 \frac{1}{12} \\ &= \frac{k}{6} + \frac{k^2}{12} \end{aligned}$$

Therefore the variance is: $var(X) = E[X^2] - E[X]^2 = \frac{k}{6} + \frac{k^2}{12}$.

Note that if we had just taken the expected value of the conditional variance $kP(1-P)$, we would have obtained $\frac{k}{6}$, which misses the other term in the total variance formula, namely the variance of the conditional mean, i.e, the variance of kP, which is $\frac{k^2}{12}$.

2. Define the following events and RVs:
W = number of hours Oscar works in a week,
T = total amount of Oscar's earnings in a week (including overtime and bonus).
Now, We want to find $\mathbf{E}[T]$ and $\text{Var}(T)$.



From the tree diagram, we use total expectation theorem,

$$\mathbf{E}[T] = \sum_{i=1}^{7} \mathbf{P}(A_i)\mathbf{E}[T|A_i]$$

Note that $\{A_1, A_2, ..., A_7\}$ is mutually exclusive and collectively exhaustive.

For each A_i, the conditional PDF $f_{T|A_i}(t)$ is constant because any linear function $aX + b$ of a uniformly distributed RV X is also uniformly distributed. Therefore,

$$\begin{aligned}
f_{T|A_1}(t) &= \tfrac{1}{50} && \text{for } 0 \le t \le 50 \\
f_{T|A_2}(t) &= \tfrac{1}{25} && \text{for } 50 < t \le 75 \\
f_{T|A_3}(t) &= \tfrac{1}{50} && \text{for } 50 < t \le 100 \\
f_{T|A_4}(t) &= \tfrac{1}{25} && \text{for } 175 < t \le 200 \\
f_{T|A_5}(t) &= \tfrac{1}{25} && \text{for } 75 < t \le 100 \\
f_{T|A_6}(t) &= \tfrac{1}{50} && \text{for } 200 < t \le 250 \\
f_{T|A_7}(t) &= \tfrac{1}{50} && \text{for } 100 < t \le 150
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}[T|A_1] &= 25 & \mathbf{E}[T|A_2] &= \tfrac{125}{2} \\
\mathbf{E}[T|A_3] &= 75 & \mathbf{E}[T|A_4] &= \tfrac{375}{2} \\
\mathbf{E}[T|A_5] &= \tfrac{175}{2} & \mathbf{E}[T|A_6] &= 225 \\
\mathbf{E}[T|A_7] &= 125.
\end{aligned}$$

Using the total expectation theorem, the expected salary per week is then equal to

$$\mathbf{E}[T] = \frac{1}{2}\cdot 25 + \frac{1}{8}\cdot\frac{125}{2} + \frac{1}{8}\cdot 75 + \frac{1}{16}\cdot\frac{375}{2} + \frac{1}{16}\cdot\frac{175}{2} + \frac{1}{16}\cdot 225 + \frac{1}{16}\cdot 125 = \boxed{68.75.}$$

For the variance of T, we need to first find $\mathbf{E}[T^2]$.

$$\mathbf{E}[T^2] = \sum_{i=1}^{7} \mathbf{P}(A_i)\mathbf{E}[T^2|A_i].$$

Using the fact that $\mathbf{E}[X^2] = (a^2 + ab + b^2)/3 = ((a+b)^2 - ab)/3$ for any uniformly distributed RV X ranging from a to b, we obtain

$$\begin{aligned}
\mathbf{E}[T^2|A_1] &= 50^2/3 & \mathbf{E}[T^2|A_2] &= (125^2 - 50\cdot 75)/3 \\
\mathbf{E}[T^2|A_3] &= (150^2 - 50\cdot 100)/3 & \mathbf{E}[T^2|A_4] &= (375^2 - 175\cdot 200)/3 \\
\mathbf{E}[T^2|A_5] &= (175^2 - 75\cdot 100)/3 & \mathbf{E}[T^2|A_6] &= (450^2 - 200\cdot 250)/3 \\
\mathbf{E}[T^2|A_7] &= (250^2 - 100\cdot 150)/3. & &
\end{aligned}$$

Therefore,

$$\mathbf{E}[T^2] = \frac{1}{2}\frac{2500}{3} + \frac{1}{8}\frac{11875}{3} + \frac{1}{8}\frac{17500}{3} + \frac{1}{16}\frac{105625}{3} + \frac{1}{16}\frac{23125}{3} + \frac{1}{16}\frac{152500}{3} + \frac{1}{16}\frac{47500}{3} = \frac{101875}{12}.$$

$$\mathrm{Var}(T) = \mathbf{E}[T^2] - (\mathbf{E}[T])^2 = \frac{180625}{48} \approx \boxed{3763.}$$