Tutorial
March 9-10, 2006

1. By definition of expected value:

$$\begin{aligned}
E[X] &= \int_0^{\infty} px\lambda e^{-\lambda x}dx + \int_{-\infty}^{0}(1-p)x\lambda e^{\lambda x}dx \\
&= p\lambda\int_0^{\infty} xe^{-\lambda x}dx + (1-p)\lambda\int_{-\infty}^{0} xe^{\lambda x}dx \\
&= p\lambda(-\frac{1}{\lambda})\left[xe^{-\lambda x}\Big|_0^{\infty} - \int_0^{\infty} e^{-\lambda x}dx\right] + (1-p)\lambda(\frac{1}{\lambda})\left[xe^{\lambda x}\Big|_{-\infty}^{0} - \int_{-\infty}^{0} e^{\lambda x}dx\right] \\
&= p\lambda(-\frac{1}{\lambda})(0-\frac{1}{\lambda}) + (1-p)\lambda(\frac{1}{\lambda})(0-\frac{1}{\lambda}) \\
&= \boxed{\tfrac{1}{\lambda}(2p-1)}
\end{aligned}$$

By definition of variance:

$$\begin{aligned}
Var(X) &= \int_0^{\infty} px^2\lambda e^{-\lambda x}dx + \int_{-\infty}^{0}(1-p)x^2\lambda e^{\lambda x}dx - (E[X])^2 \\
&= p\frac{2}{\lambda^2} + (1-p)\frac{2}{\lambda^2} - \frac{1}{\lambda^2}(2p-1)^2 \\
&= \boxed{\tfrac{2}{\lambda^2} - \tfrac{1}{\lambda^2}(2p-1)^2}
\end{aligned}$$

Another way of finding the expectation and the variance:

Let A be the event such that $x > 0$. Using the Total Probability Theorem:

$$\begin{aligned}
E[X] &= P(A)E[X|A] + P(A^c)E[X|A^c] \\
&= p * (\frac{1}{\lambda} + (1-p) * (-\frac{1}{\lambda}) \\
&= \boxed{\tfrac{1}{\lambda}(2p-1)}
\end{aligned}$$

For variance, we use the formula:

$$\begin{aligned}
Var(X) &= E[X^2] - (E[X])^2 \\
&= P(A)E[X^2|A] + P(A^c)E[X^2|A^c] - (E[X])^2
\end{aligned}$$

The value for $E[X^2|A]$ can be computed as follows:

$$\begin{aligned} Var(X|A) &= E[X^2|A] - (E[X|A])^2 \\ \frac{1}{\lambda^2} &= E[X^2|A] - (\frac{1}{\lambda})^2 \\ E[X^2|A] &= \frac{2}{\lambda^2} \end{aligned}$$

We can find $E[X^2|A^c]$ following the same logic. Let's continue with computing variance using the values for $E[X^2|A]$ and $E[^2|A^c]$.

$$\begin{aligned} Var(X) &= P(A)E[X^2|A] + P(A^c)E[X^2|A^c] - (E[X])^2 \\ &= p * \frac{2}{\lambda^2} + (1-p)(\frac{2}{\lambda^2}) - \frac{1}{\lambda^2}(2p-1)^2 \\ &= \boxed{\frac{2}{\lambda^2} - \frac{1}{\lambda^2}(2p-1)^2} \end{aligned}$$

2. (a) Let G represent the event that the weather is good. We are given $\mathbf{P}(G) = \frac{2}{3}$. To find the PDF of X, we first find the PDF of W, since $X = s + W = 2 + W$. We know that given good weather, $W \sim N(0, 1)$. We also know that given bad weather, $W \sim N(0, 9)$. To find the unconditional PDF of W, we use the density version of the total probability theorem.

$$\begin{aligned} f_W(w) &= \mathbf{P}(G) \cdot f_{W|G}(w) + \mathbf{P}(G^c) \cdot f_{W|G^c}(w) \\ &= \frac{2}{3} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} + \frac{1}{3} \cdot \frac{1}{3\sqrt{2\pi}} e^{-\frac{w^2}{2(9)}} \end{aligned}$$

We now perform a change of variables using $X = 2 + W$ to find the PDF of X:

$$f_X(x) = f_W(x-2) = \frac{2}{3} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2}} + \frac{1}{3} \cdot \frac{1}{3\sqrt{2\pi}} e^{-\frac{(x-2)^2}{18}}.$$

(b) In principle, one can use the PDF determined in part (a) to compute the desired probability as

$$\int_1^3 f_X(x)\, dx.$$

It is much easier, however, to translate the event $\{1 \le X \le 3\}$ to a statement about W and then to apply the total probability theorem.

$$\mathbf{P}(1 \le X \le 3) = \mathbf{P}(1 \le 2 + W \le 3) = \mathbf{P}(-1 \le W \le 1)$$

We now use the total probability theorem.

$$\mathbf{P}(-1 \le W \le 1) = \mathbf{P}(G) \underbrace{\mathbf{P}(-1 \le W \le 1 \mid G)}_{a} + \mathbf{P}(G^c) \underbrace{\mathbf{P}(-1 \le W \le 1 \mid G^c)}_{b}$$

Since conditional on either G or G^c the random variable W is Gaussian, the conditional probabilities a and b can be expressed using Φ. Conditional on G, we have $W \sim N(0,1)$ so

$$a = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1.$$

Conditional on G^c, we have $W \sim N(0,9)$ so

$$b = \Phi\left(\frac{1}{3}\right) - \Phi\left(-\frac{1}{3}\right) = 2\Phi\left(\frac{1}{3}\right) - 1.$$

The final answer is thus

$$\mathbf{P}(1 \leq X \leq 3) = \frac{2}{3}\left(2\Phi\left(1\right) - 1\right) + \frac{1}{3}\left(2\Phi\left(\frac{1}{3}\right) - 1\right).$$

3.

(a) Using the total expectation theorem, we obtain
$\mathbf{E}[X] = \mathbf{E}[X|A]\mathbf{P}(A) + \mathbf{E}[X|B]\mathbf{P}(B) = 1 * \frac{1}{2} + \frac{1}{3} * \frac{1}{2} = \frac{2}{3}$

(b) Using the total probability theorem, we obtain
$\mathbf{P}(D) = \mathbf{P}(D|A)\mathbf{P}(A) + \mathbf{P}(D|B)\mathbf{P}(B) = \frac{1}{2}e^{-\tau} + \frac{1}{2}e^{-3\tau}$

(c) Using the Bayes' theorem, we obtain
$\mathbf{P}(T_{1A}|D) = \frac{\mathbf{P}(D|T_{1A})\mathbf{P}(T_{1A})}{\mathbf{P}(D)} = \frac{1}{1+e^{-2\tau}}$

(d) Using the total expectation theorem, we obtain
$\mathbf{E}$[Total Time Till Failure | D]
$= \tau + \mathbf{E}[\text{Time to failure after } \tau\ |D,A]\mathbf{P}(A|D) + \mathbf{E}[\text{Time to failure after } \tau\ |D,B]\mathbf{P}(B|D)$
$= \tau + \frac{1}{1+e^{-2\tau}} + (\frac{1}{3})\frac{e^{-2\tau}}{1+e^{-2\tau}}$