Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

Recitation 09 Solutions
March 21, 2006

1.

(a) $P[A] = \frac{7}{8}$

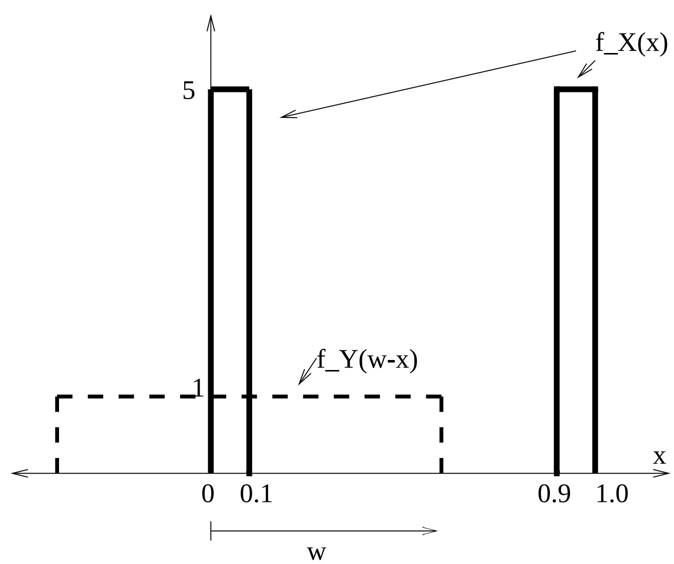
(b) $P[\text{Al wins 7 out of 10 races}] = \binom{10}{7}(\frac{7}{8})^7(\frac{1}{8})^3$

(c) $f_w(w_0) = \begin{cases} \frac{1}{2}, & 1 < w_0 \leq 2 \\ \frac{7}{4} - \frac{w_0}{2}, & 2 < w_0 \leq 3 \\ 0, & \text{otherwise} \end{cases}$

2.

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)dx$$

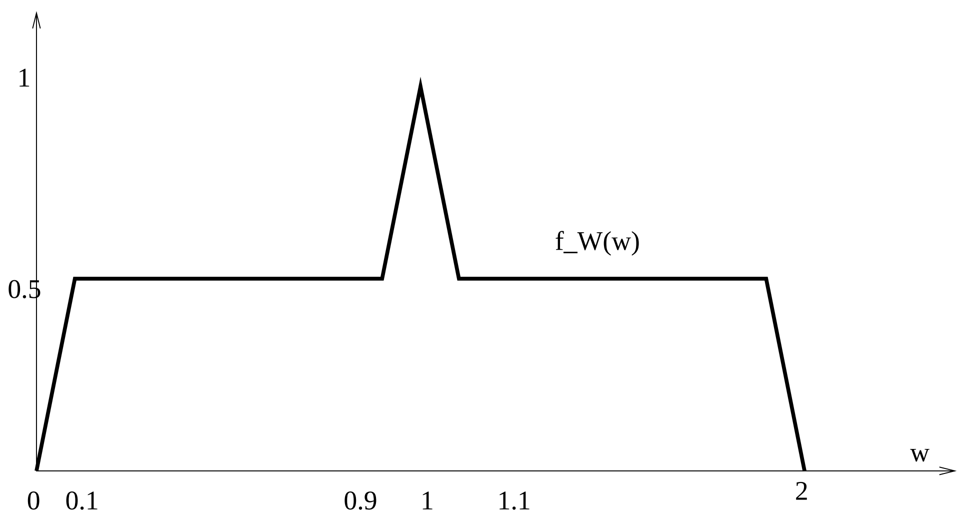
for $w = x + y$ and x, y independent. This operation is called the *convolution* of $f_X(x)$ and $f_Y(y)$. Graphically, $f_Y(w-x)$ is obtained by "flipping" $f_Y(x)$ (note that we are plotting the pdf for Y as a function of x at this point) about the $x = 0$ axis, then shifting that plot to the right by w. $f_X(x)$ is then sketched on the same plot.



From this graph we compute the integral of the product of the curves as a function of w. By visualizing the graph as w is varied, we obtain

$$f_W(w) = \begin{cases} 5w, & 0 \le w \le 0.1 \\ 0.5, & 0.1 \le w \le 0.9 \\ 5(0.1 + (w - 0.9)), & 0.9 \le w \le 1.0 \\ 5(0.1 + (1.1 - w)), & 1.0 \le w \le 1.1 \\ 0.5, & 1.1 \le w \le 1.9 \\ 5(2.0 - w), & 1.9 \le w \le 2.0 \\ 0, & \text{otherwise} \end{cases}$$

Pictorially,



3. Let X and Y be the number of flips until Alice and Bob stop, respectively. Thus, $X + Y$ is the total number of flips until both stop. The random variables X and Y are independent geometric random variables with parameters 1/4 and 3/4, respectively. By convolution, we have

$$\begin{aligned} p_{X+Y}(j) &= \sum_{k=-\infty}^{\infty} p_X(k) p_Y(j-k) \\ &= \sum_{k=1}^{j-1} (1/4)(3/4)^{k-1}(3/4)(1/4)^{j-k-1} \\ &= \frac{1}{4^j} \sum_{k=1}^{j-1} 3^k \\ &= \frac{1}{4^j} \left(\frac{3^j - 1}{3 - 1} - 1 \right) \end{aligned}$$

$$= \frac{3}{2}\frac{(3^{j-1}-1)}{4^j},$$

if $j \geq 2$, and 0 otherwise. (Even though $X+Y$ is *not* geometric, it roughly behaves like one with parameter $3/4$.)