Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

Recitation 08 Solutions
March 09, 2006

1. In order to solve this problem, it is useful to begin by deriving the marginal distributions $f_X(x)$ and $f_Y(y)$. The marginal distributions are obtained by integrating the joint distribution along the X and Y axes and is shown in the following figure.

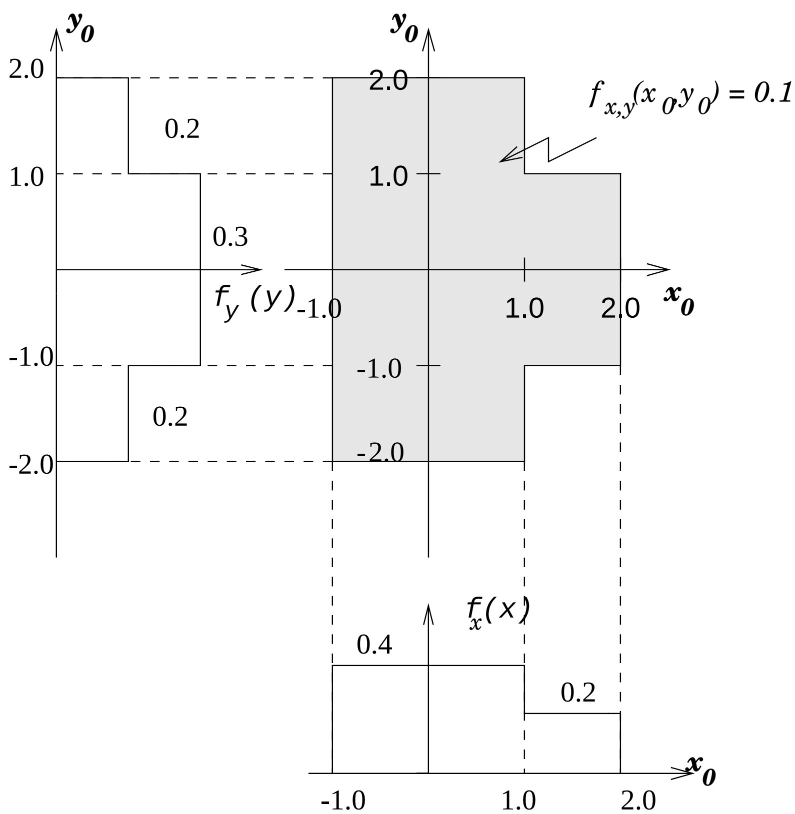

Figure 1: Marginal probabilities $f_X(x)$ and $f_Y(y)$ obtained by integration along the y and x axes respectively

(a) The conditional PDF $f_{Y|X}(y|x)$ is given by

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

A good way to visualize the conditional PDF $f_{Y|X}(y|x)$ is to imagine a vertical slice of the joint PDF at $X = x$. Essentially, the conditional PDF has the same shape as the joint PDF except for a scaling factor $f_X(x)$ which ensures that,

$$\int f_{Y|X}(y|x)dy = 1$$

Similarly the conditional PDF $f_{X|Y}(x|y)$ is obtained using,

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

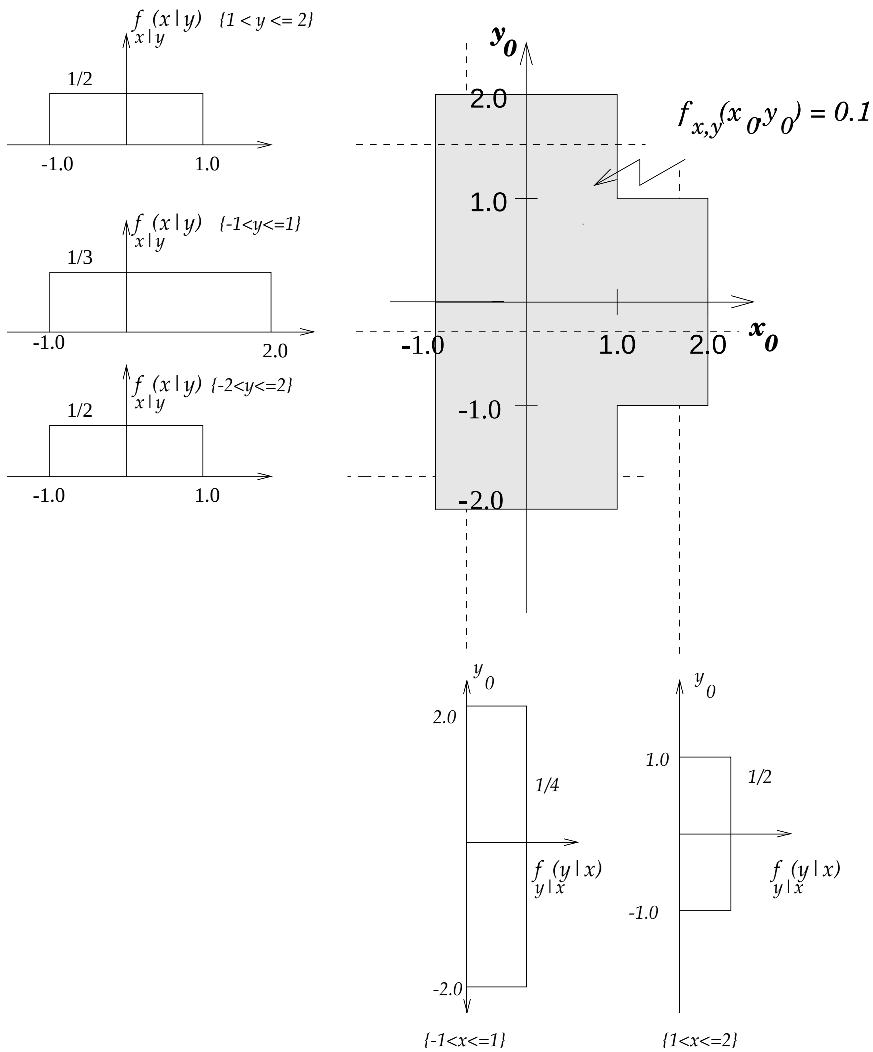

Figure 2: Marginal probabilities $f_X(x)$ and $f_Y(y)$ obtained by integration along the y and x axes respectively

To visualize $f_{X|Y}(x|y)$, imagine a horizontal slice through the joint PDF at$Y = y$. Again, the conditional PDF has the same shape except for the scaling factor of $f_y(y_0)$. The conditional PDFs are as shown in the figure below.

(b) X and Y are **NOT** indepenent since $f_{XY}(x,y) \neq f_X(x)f_Y(y)$. Also, from the figures we have $f_{X|Y}(x|y) \neq f_X(x)$ and $f_{Y|X}(y|x) \neq f_Y(y)$.

(c)

$$\begin{aligned} f_{XY}(x,y|A) &= \frac{f_{XY}((x,y) \cap (x,y) \in A)}{\mathbf{P}(A)} \\ &= \left\{ \begin{array}{ll} \frac{0.1}{0.1\pi} & (x,y) \in A \\ 0 & \text{otherwise} \end{array} \right\} \end{aligned}$$

(d) It can be seen that $f_{X|Y}(x|y)$ is a uniform density for each value of y. Noting the fact that the expected value of a random variable distributed uniformly between a and b is given by $\frac{(a+b)}{2}$, the conditional expected values can be written as

$$\mathbf{E}[X|Y] = \left\{ \begin{array}{ll} 0 & -2.0 \leq Y \leq -1.0 \\ \frac{1}{2} & -1.0 \leq Y \leq 1.0 \\ 0 & 1.0 \leq Y \leq 2.0 \end{array} \right\}$$

Again, noting the fact that the variance of a random variable distributed uniformly between a and b is given by $\frac{(b-a)^2}{12}$, the conditional variance $\sigma^2_{X|Y}$ is given by

$$\sigma^2_{X|Y} = \left\{ \begin{array}{ll} \frac{4}{12} & -2.0 \leq Y \leq -1.0 \\ \frac{9}{12} & -1.0 \leq Y \leq 1.0 \\ \frac{4}{12} & 1.0 \leq Y \leq 2.0 \end{array} \right\}$$

The expected value $\mathbf{E}[X]$ can be obtained by averaging the conditional mean over y.

$$\begin{aligned} \mathbf{E}[x] = \mathbf{E}[\mathbf{E}[x|y]] &= (0.0)\mathbf{P}(-2 \leq Y \leq -1) + (0.5)\mathbf{P}(-1 \leq Y \leq 1) + (0.0)\mathbf{P}(1 \leq Y \leq 2) \\ &= (0.5)(0.6) = 0.3 \end{aligned}$$

(e) The conditional expected value $\mathbf{E}[Y|X]$ and the conditional variance $\sigma_{Y|X}$ is computed in a similar fashion

$$\mathbf{E}[Y|X] = \left\{ \begin{array}{ll} 0 & -1.0 \leq X \leq 1 \\ 0 & 1.0 \leq X \leq 2.0 \end{array} \right\}$$

$$\sigma^2_{Y|X} = \left\{ \begin{array}{ll} \frac{16}{12} & -1.0 \leq X \leq 1.0 \\ \frac{4}{12} & 1.0 \leq X \leq 2.0 \end{array} \right\}$$

The expected value $\mathbf{E}[Y]$ can be obtained by averaging the conditional mean over x.

$$\begin{aligned} \mathbf{E}[y] = \mathbf{E}[\mathbf{E}[Y|X]] &= (0.0)\mathbf{P}(-1 \leq X \leq 1) + (0.0)\mathbf{P}(1 \leq X \leq 2) \\ &= (0)(0.8 + (0)(0.2)) = 0.0 \end{aligned}$$

(f) It is to be noted that σ^2_X cannot be simply obtained by evaluating $E[\sigma^2_{X|Y}]$ as we do with the means. By definition of variance,

$$\begin{aligned} \sigma^2_X &= \mathbf{E}[X^2] - (\mathbf{E}[X])^2 \\ &= \mathbf{E}[\mathbf{E}[X^2|Y]] - (\mathbf{E}[\mathbf{E}[X|Y]])^2 \\ &= \mathbf{E}[\sigma^2_{X|Y} + \mathbf{E}[X|Y]^2) - (\mathbf{E}[X])^2 \end{aligned}$$

We have already computed both $\sigma^2_{X|Y}$ and $\mathbf{E}[x|y]^2$. Therefore, we obtain,

$$\begin{aligned} \sigma^2_X &= (\frac{4}{12})\mathbf{P}(-2 \leq Y \leq -1) + (\frac{9}{12} + (\frac{1}{2})^2)\mathbf{P}(-1 \leq Y \leq 1) + (\frac{4}{12})P(1 \leq Y \leq 2) - (\mathbf{E}[x])^2 \\ &= 0.643 \end{aligned}$$

Similarly, we can get $\sigma^2_Y = 1.132$.

2. See online supplementary problem solutions (Problem 14, Section 3.5)