Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2006)

Tutorial 06 Solutions
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1. (a) Using the information in the problem and moment generating properties $M_X(0) = 1$ and $\frac{d}{ds}M_X(s)\Big|_{s=0} = E[X]$, we obtain a system of two equations for a and b:

$$M_X(0) = ae^0 + be^{13(e^0-1)} = 1 \Rightarrow a + b = 1;$$

$$E[X] = 3 = ae^0 + 13be^0e^{13(e^0-1)} \Rightarrow a + 13b = 5.$$

Therefore,

$$a = \frac{2}{3}, \quad b = \frac{1}{3}.$$

(b) $E[e^{5X}] = M_X(s)\Big|_{s=5} = ae^5 + be^{13(e^5-1)} = \frac{2}{3}e^5 + \frac{1}{3}e^{13(e^5-1)} = 6.20 \times 10^{831}$.

(c) One way to solve this part is to find the PDF of X and then find $P(X = 1)$.

We will demonstrate an alternative way of computing $P(X = 1)$ directly from the transform. This is useful if one cannot invert the transform and obtain the PDF directly.

It is easy to see that X is a discrete random variable (its transform is a combination of transforms of two PMFs). Moreover, X takes on nonnegative values (either 1 or the values of a Poisson random variable). There is a fact about the transform of a nonnegative discrete random variable which will be very useful here:

$$\frac{d^n}{d(e^s)^n}M_X(s)\Big|_{e^s=0} = n!p_X(n).$$

To see this, note that for a nonnegative discrete random variable X we can write

$$M_X(s) = E[e^{sX}] = p_X(0)\cdot e^{0s} + p_X(1)\cdot e^{1s} + p_X(2)\cdot e^{2s} + ...$$

Then,

$$M_X(s)\Big|_{e^s=0} = p_X(0),$$

$$\frac{d}{de^s}M_X(s)\Big|_{e^s=0} = (p_X(1)\cdot 1 + 2p_X(2)e^s + ...)\Big|_{e^s=0} = p_X(1),$$

$$\frac{d^2}{d(e^s)^2}M_X(s)\Big|_{e^s=0} = (2!p_X(2) + 3!p_X(3)e^s + ...)\Big|_{e^s=0} = 2!p_X(2),$$

etc. Therefore,

$$P(X=1) = p_X(1) = \frac{1}{1!}\left(ae^s + be^{13(e^s-1)}\right)'\Big|_{e^s=0} = \left(a + 13be^{13(e^s-1)}\right)\Big|_{e^s=0} =$$

$$= a + 13be^{-13} = \frac{2}{3} + 13\cdot\frac{1}{3}\cdot e^{-13} = 0.667.$$

(d) $E[X^2] = \frac{d^2}{ds^2}M(s)\Big|_{s=0} = \Big(ae^s + 13b(e^s e^{13(e^s-1)} + e^s \cdot 13e^s \cdot e^{13(e^s-1)})\Big)\Big|_{s=0}$

$$= a + 182b = \frac{184}{3}.$$

2. (a) $\mathbf{E}[e^{s(5Z+1)}] = e^s\mathbf{E}[e^5 sz] = e^s M_Z(5s) = e^{s+5(e^{5s}-1)}$.

   (b) $M_{X+Y}(s) = M_X(s)M_Y(s) = (\frac{3}{4} + \frac{1}{4}e^s)\frac{3}{3-s}$. Note that s must be less that 3 for this to hold.

   (c) $\mathbf{E}[e^{s(XY+(1-X)Z)}] = p_X(0)\mathbf{E}[e^{sZ}|X = 0] + p_X(1)\mathbf{E}[e^{sY}|X = 1] = \frac{3}{4}M_Z(s) + \frac{1}{4}M_Y(s) = \frac{3}{4}e^{5(e^s-1)} + \frac{1}{4}\frac{3}{3-s}$. Note that s must be less that 3 for this to hold.

3. If $3 \le z \le 6$, we have

$$\begin{aligned} f_{X+Y}(z) &= \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\,dx \\ &= \int_{\max(0,z-4)}^{\min(2,z-3)} \frac{1}{2}\,dx \\ &= (\min(2,z-3) - \max(0,z-4))/2. \end{aligned}$$

The PDF of $X + Y$ is then

$$f_{X+Y}(z) = \begin{cases} \frac{z-3}{2} & 3 \le z < 4 \\ \frac{1}{2} & 4 \le z < 5 \\ \frac{6-z}{2} & 5 \le z \le 6 \\ 0 & \text{otherwise} \end{cases}$$

The sketch has been omitted.