Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

Tutorial 3: Solutions
March 2-3, 2006

1. (a)

$$p_S(s) = \sum_r \mathbf{P}(R = r,\ S = s)$$

$$p_{S|A}(s) = \frac{\mathbf{P}(S = s,\ A)}{\mathbf{P}(A)} = \frac{\mathbf{P}(S = s,\ S \neq 3)}{\mathbf{P}(S \neq 3)}$$

Plug in the values from the figure, we get the following answers:

$$p_S(s) = \begin{cases} 24/90, & s = 1; \\ 36/90, & s = 2; \\ 30/90, & s = 3; \\ 0, & \text{otherwise,} \end{cases}$$

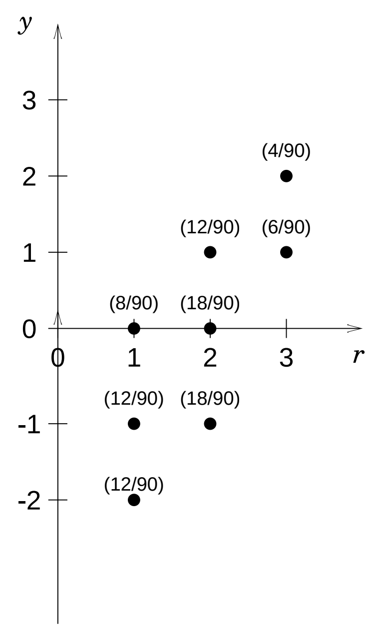
$$p_{S|A}(s) = P(S = s \cap A)/P(A) = \begin{cases} 24/60, & s = 1; \\ 36/60, & s = 2; \\ 0, & \text{otherwise.} \end{cases}$$

(The required sketches are omitted.)

(b) The joint PMF for R and Y can be calculated as follows:

$$p_{R,Y}(r,\ y) = p_R(r)\mathbf{P}(R - S = y|R = r) = p_R(r)\mathbf{P}(S = r - y|R = r) = p_{R,S}(r,\ r - y)$$

We sketch the joint PMF for R and Y in the following figure:



(c)

$$p_{X|A}(x) = P(X = x \cap A)/P(A) = \begin{cases} 8/60, & x = 2; \\ 24/60, & x = 3; \\ 22/60, & x = 4; \\ 6/60, & x = 5; \\ 0, & \text{otherwise.} \end{cases}$$

(The required sketch is omitted.)

2. We are given the following information:

$$p_K(k) = \begin{cases} 1/4, & \text{if } k = 1, 2, 3, 4; \\ 0, & \text{otherwise} \end{cases}$$

$$p_{N|K}(n \mid k) = \begin{cases} 1/k, & \text{if } n = 1, \ldots, k \\ 0, & \text{otherwise} \end{cases}$$

(a) We use the fact that $p_{N,K}(n, k) = p_{N|K}(n \mid k)p_K(k)$ to arrive at the following joint PMF:

$$p_{N,K}(n, k) = \begin{cases} 1/(4k), & \text{if } k = 1, 2, 3, 4 \text{ and } n = 1, \ldots, k \\ 0, & \text{otherwise} \end{cases}$$

(b) The marginal PMF $p_N(n)$ is given by the following formula:

$$p_N(n) = \sum_k p_{N,K}(n, k) = \sum_{k=n}^{4} \frac{1}{4k}$$

On simplification this yields

$$p_N(n) = \begin{cases} 1/4 + 1/8 + 1/12 + 1/16 = 25/48, & n = 1; \\ 1/8 + 1/12 + 1/16 = 13/48, & n = 2; \\ 1/12 + 1/16 = 7/48, & n = 3; \\ 1/16 = 3/48, & n = 4; \\ 0, & \text{otherwise.} \end{cases}$$

(c) The conditional PMF is

$$p_{K|N}(k \mid 2) = \frac{p_{N,K}(2, k)}{p_N(2)} = \begin{cases} 6/13, & k = 2; \\ 4/13, & k = 3; \\ 3/13, & k = 4; \\ 0, & \text{otherwise.} \end{cases}$$

(d) Let A be the event $2 \le N \le 3$. We first find the conditional PMF of K given A.

$$\begin{aligned}
p_{K|A}(k) &= \frac{\mathbf{P}(K=k, A)}{\mathbf{P}(A)} \\
\mathbf{P}(A) &= p_N(2) + p_N(3) = \frac{5}{12} \\
\mathbf{P}(K=k, A) &= \begin{cases} \frac{1}{8}, & k=2; \\ \frac{1}{12} + \frac{1}{12}, & k=3; \\ \frac{1}{16} + \frac{1}{16}, & k=4; \\ 0, & \text{otherwise} \end{cases} \\
p_{K|A}(k) &= \begin{cases} \frac{3}{10}, & k=2; \\ \frac{2}{5}, & k=3; \\ \frac{3}{10}, & k=4; \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

Because the conditional PMF of K given A is symmetric around $k = 3$, we know $\boxed{\mathbf{E}[K \mid A] = 3}$. We now find the conditional variance of K given A.

$$\begin{aligned}
var(K \mid A) &= \mathbf{E}[(K - \mathbf{E}[K \mid A])^2 \mid A] \\
&= \frac{3}{10} \cdot (2-3)^2 + \frac{2}{5} \cdot 0 + \frac{3}{10} \cdot (4-3)^2 \\
&= \boxed{\tfrac{3}{5}}
\end{aligned}$$

(e) Let C_i be the cost of book i and $\mathbf{E}[C_i] = 3$. Let T be the total cost, so $T = C_1 + \ldots + C_N$. We now find $\mathbf{E}[T]$ using the total expectation theorem.

$$\begin{aligned}
\mathbf{E}[T] &= \mathbf{E}[T \mid N=1]p_N(1) + \mathbf{E}[T \mid N=2]p_N(2) + \mathbf{E}[T \mid N=3]p_N(3) + \mathbf{E}[T \mid N=4]p_N(4) \\
&= \mathbf{E}[C_1]p_N(1) + \mathbf{E}[C_1 + C_2]p_N(2) + \mathbf{E}[C_1 + C_2 + C_3]p_N(3) + \mathbf{E}[C_1 + C_2 + C_3 + C_4]p_N(4) \\
&= \mathbf{E}[C_i]p_N(1) + 2\mathbf{E}[C_i]p_N(2) + 3\mathbf{E}[C_i]p_N(3) + 4\mathbf{E}[C_i]p_N(4) \\
&= 3 \cdot \frac{25}{48} + 6 \cdot \frac{13}{48} + 9 \cdot \frac{7}{48} + 12 \cdot \frac{1}{16} \\
&= \boxed{\tfrac{21}{4}}
\end{aligned}$$

3. (a) An easy way to derive $p_{X,Y,Z}(x,y,z)$ is in sequential terms as $p_X(x) \cdot p_{Y,Z|X}(y,z|x)$. Note $p_X(x)$ is geometric with parameter p. Conditioned on X even, $(Y,Z) = (0,0)$ with probability 1. Conditioned on X odd, $p_{Y,Z|X}(y,z) = \frac{1}{4}$ for $(y,z) \in \{(0,0),(0,2),(2,0),(2,2)\}$.

$$p_{X,Y,Z}(x,y,z) = \begin{cases} \frac{1}{4}p(1-p)^{x-1}, & \text{if } x \text{ is odd and } (y,z) \in \{(0,0),(0,2),(2,0),(2,2)\} \\ p(1-p)^{x-1}, & \text{if } x \text{ is even and } (y,z) = (0,0) \\ 0, & \text{otherwise.} \end{cases}$$

(b) (i) No. Notice that even though conditional on X (i.e. given a realization, x, of random variable X), the random variables Y and Z are independent (that's why they look "regular"), Y and Z are not independent. Given Y, the distribution over Z changes (i.e. if Y is 2, Z is equally likely to be 0 or 2; however if Y is 0, Z is more likely to be 0).

(ii) Yes. Given $Z = 2$, if we are further given $X = x$, Y is equally likely to take on the value 0 or 2.

(iii) No. Given $Z = 0$, if we are further given $X = x$, then if x is even, Y must be 0, whereas if x is odd, Y is equally likely to take on 0 or 2.

(iv) Yes. Given $Z = 2$, if we are further given $X = x$, $Z = 2$ still holds (i.e. with probability 1)! Double conditioning has no effect.

(c) If $X = 5$, then Y and Z are uniformly distributed on the set S specified in the problem statement, so $Y + Z$ takes the values 0 and 4 with probability $\frac{1}{4}$, and takes the value 2 with probability $\frac{1}{2}$. This PMF is symmetric about 2, so the mean value of $Y + Z$ is evidently 2. Hence the variance is

$$(0-2)^2\frac{1}{4} + (4-2)^2\frac{1}{4} = 2.$$