Final Exam

YOUR NAME: _____

:

- You may use **two** $8.5 \times 11''$ sheets with notes in your own handwriting on both sides, but no other reference materials. Calculators are not allowed.

- You may assume all results presented in class.

- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.

- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.

GOOD LUCK!

Problem	Points	Grade	Grader
1	13		
2	15		
3	12		
4	12		
5	12		
6	12		
7	12		
8	12		
Total	100		

Problem 1. [13 points] Give an inductive proof that the Fibonacci numbers F_n and F_{n+1} are relatively prime for all $n \geq 0$. The Fibonacci numbers are defined as follows:

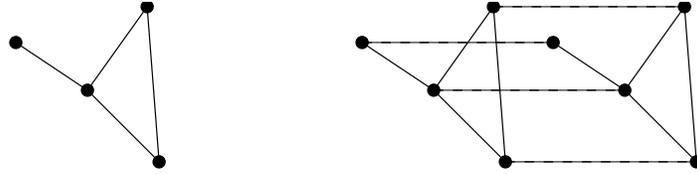$$F_0 = 0 \qquad F_1 = 1 \qquad F_n = F_{n-1} + F_{n-2} \quad \text{(for } n \geq 2\text{)}$$

Solution. We use induction on n. Let $P(n)$ be the proposition that F_n and F_{n+1} are relatively prime.

Base case: $P(0)$ is true because $F_0 = 0$ and $F_1 = 1$ are relatively prime.

Inductive step: Assume that $P(n)$ is true where $n \geq 0$; that is, F_n and F_{n+1} are relatively prime. We must show that F_{n+1} and F_{n+2} are relatively prime as well. If F_{n+1} and F_{n+2} had a common divisor $d > 1$, then d would also divide the linear combination $F_{n+2} - F_{n+1} = F_n$, contradicting the assumption that F_n and F_{n+1} are relatively prime. So F_{n+1} and F_{n+2} are relatively prime.
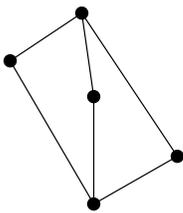
The theorem follows by induction.

Problem 2. [15 points] The *double* of a graph G consists of two copies of G with edges joining corresponding vertices. For example, a graph appears below on the left and its double appears on the right.
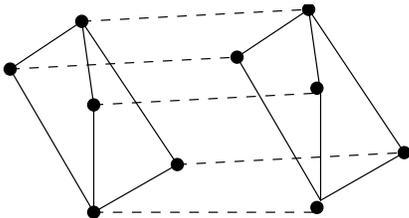
Some edges in the graph on the right are dashed to clarify its structure.

(a) Draw the double of the graph shown below.

Solution.

(b) Suppose that G_1 is a bipartite graph, G_2 is the double of G_1, G_3 is the double of G_2, and so forth. Use induction on n to prove that G_n is bipartite for all $n \geq 1$.

Solution. We use induction. Let $P(n)$ be the proposition that G_n is bipartite.

Base case: $P(1)$ is true because G_1 is bipartite by assumption.

Inductive step: For $n \geq 1$, we assume $P(n)$ in order to prove $P(n + 1)$. The graph G_{n+1} consists of two subgraphs isomorphic to G_n with edges joining corresponding vertices. Remove these extra edges. By the assumption $P(n)$, we can color each vertex of one subgraph black or white so that adjacent vertices get different colors. If we color the corresponding vertices in the other subgraph oppositely, then adjacent vertices get different colors within that subgraph as well. And now if we add back the extra edges, each of these joins a white vertex and a black vertex. Therefore, G_{n+1} is bipartite.
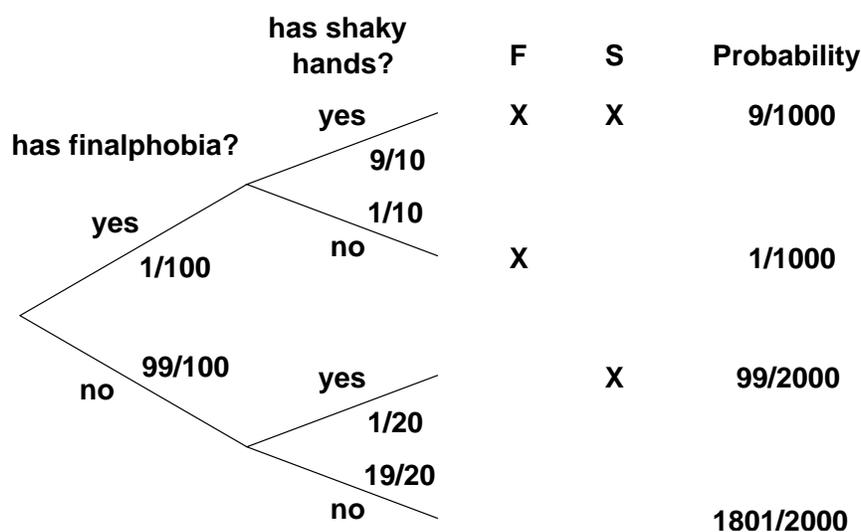
The theorem follows from the principle of induction.

Problem 3. [12 points] *Finalphobia* is a rare disease in which the victim has the delusion that he or she is being subjected to an intense mathematical examination.

- A person selected uniformly at random has finalphobia with probability $1/100$.

- A person with finalphobia has shaky hands with probability $9/10$.

- A person without finalphobia has shaky hands with probability $1/20$.

What is the probablility that a person selected uniformly at random has finalphobia, given that he or she has shaky hands?

Solution. Let F be the event that the randomly-selected person has finalphobia, and let S be the event that he or she has shaky hands. A tree diagram is worked out below:

The probability that a person has finalphobia, given that he or she has shaky hands is:

$$\Pr\left(F \mid S\right) = \frac{\Pr\left(F \cap S\right)}{\Pr\left(S\right)}$$
$$= \frac{9/1000}{9/1000 + 99/2000}$$
$$= \frac{18}{18 + 99}$$
$$= \frac{18}{117}$$

Problem 4. [12 points] Suppose that you roll five 6-sided dice that are fair and mutually independent. For the problems below, answers alone are sufficient, but we can award partial credit only if you show your work. Also, you do not need to simplify your answers; you may leave factorials, binomial coefficients, and arithmetic expressions unevaluated.

(a) What is the probability that all five dice show different values?

Example: $(1, 2, 3, 4, 5)$ is a roll of this type, but $(1, 1, 2, 3, 4)$ is not.

Solution. The probability space is the uniform distribution on the 6^5 possible numbers rolled on the five (distinguishable) dice. So the probability that all dice are different is the number of outcomes in which the dice have distinct values divided by 6^5. There are $(6)_5 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$ such outcomes so

$$\Pr\left(\text{all rolls distinct}\right) = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{6^5} = \frac{5!}{6^4}$$

An alternative approach uses the observation that the conditional probability that the $i + 1$st die value differs from the preceding rolls, given that the first i values differ, is $(6 - i)/6$ for $1 \le i \le 4$, and the probability that all five values are different is the product of these conditional probabilities, namely

$$\Pr\left(\text{all rolls distinct}\right) = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} = \frac{5!}{6^4}$$

(b) What is the probability that two dice show the same value and the remaining three dice all show different values?

Example: $(6, 1, 6, 2, 3)$ is a roll of this type, but $(1, 1, 2, 2, 3)$ and $(4, 4, 4, 5, 6)$ are not.

Solution. There are $\binom{5}{2}$ possible pairs of rolls that might have the same value and 6 possibilities for what this value is. There $5 \cdot 4 \cdot 3$ possible distinct values for the remaining three rolls. So

$$\Pr\left(\text{exactly two values the same}\right) = \frac{\binom{5}{2} \cdot 6 \cdot 5 \cdot 4 \cdot 3}{6^5} = \frac{100}{6^3}$$

An alternative way to count is: there are $\binom{6}{4}$ sets of four values among the five dice, 4 choices for which of these values is repeated, and by the Bookkeeper rule, $\binom{5}{2,1,1,1} = 5!/2$ permutations of a sequence consisting of five values, one of which appears twice. So,

$$\Pr\left(\text{exactly two values the same}\right) = \frac{\binom{6}{4} \cdot 4 \cdot 5!/2}{6^5} = \frac{100}{6^3}$$

(c) What is the probability that two dice show one value, two different dice show a second value, and the remaining die shows a third value?

Example: $(6, 1, 2, 1, 2)$ is a roll of this type, but $(4, 4, 4, 4, 5)$ and $(5, 5, 5, 6, 6)$ are not.
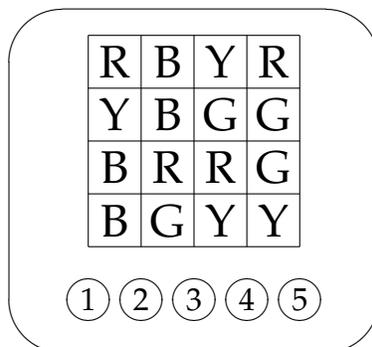
Solution. There are $\binom{6}{2}$ sets of two values that might be duplicated. There are $\binom{5}{2}$ rolls where larger duplicated value may come up and $\binom{3}{2}$ remaining rolls where the smaller duplicated value may come up. There is only 1 remaining roll where the nonduplicated value may then come up, and 4 remaining values it could take. So,

$$\Pr\left(\text{exactly two pairs of same values}\right) = \frac{\binom{6}{2} \cdot \binom{5}{2} \cdot \binom{3}{2} \cdot 4}{6^5} = \frac{50}{6^3}$$

Alternatively, there are $\binom{6}{3}$ sets of three values among the five dice, 3 choices for which of these values is not repeated, and by the Bookkeeper rule, $\binom{5}{2,2,1}$ permutations of a sequence consisting of five values, two of which appear twice. So,

$$\Pr\left(\text{exactly two pairs of same values}\right) = \frac{\binom{6}{3} \cdot 3 \cdot \binom{5}{2,2,1}}{6^5} = \frac{50}{6^3}$$

Problem 5. [12 points] An electronic toy displays a 4×4 grid of colored squares. At all times, four are red, four are green, four are blue, and four are yellow. For example, here is one possible configuration:

For parts (a) and (b) below, you need not simplify your answers.

    (a) How many such configurations are possible?

    Solution. This is equal to the number of sequences containing 4 R's, 4 G's, 4 B's, and 4 Y's, which is:
$$\frac{16!}{(4!)^4}$$

    (b) Below the display, there are five buttons numbered 1, 2, 3, 4, and 5. The player may press a sequence of buttons; however, the same button can not be pressed twice in a row. How many different sequences of n button-presses are possible?

    Solution. There are 5 choices for the first button press and 4 for each subsequence press. Therefore, the number of different sequences of n button presses is $5 \cdot 4^{n-1}$.

    (c) Each button press scrambles the colored squares in a complicated, but nonrandom way. Prove that there exist two *different* sequences of 32 button presses that both produce the *same* configuration, if the puzzle is initially in the state shown above. (Hint: $4^{32} = 16^{16} > 16!$)

    Solution. We use the Pigeonhole Principle. Let A be the set of all sequences of 32 button presses, let B be the set of all configurations, and let $f : A \to B$ map each sequence of button presses to the configuration that results. Now:
$$|A| > 4^{32} > 16! > |B|$$

Thus, by the Pigeonhole Principle, f is not injective; that is, there exist distinct elements $a_1, a_2 \in A$ such taht $f(a_1) = f(a_2)$. In other words, there are two different sequences of button presses that produce the same configuration.

Problem 6. [12 points] MIT students sometimes delay laundry for a few days. Assume all random values described below are mutually independent.

  (a) A *busy* student must complete 3 problem sets before doing laundry. Each problem set requires 1 day with probability $2/3$ and 2 days with probability $1/3$. Let B be the number of days a busy student delays laundry. What is $\operatorname{Ex}(B)$?

  Example: If the first problem set requires 1 day and the second and third problem sets each require 2 days, then the student delays for $B = 5$ days.

  Solution. The expected time to complete a problem set is:

  $$1 \cdot \frac{2}{3} + 2 \cdot \frac{1}{3} = \frac{4}{3}$$

  Therefore, the expected time to complete all three problem sets is:

  $$\operatorname{Ex}(B) = \operatorname{Ex}(\text{pset1}) + \operatorname{Ex}(\text{pset2}) + \operatorname{Ex}(\text{pset3})$$
  $$= \frac{4}{3} + \frac{4}{3} + \frac{4}{3}$$
  $$= 4$$

  (b) A *relaxed* student rolls a fair, 6-sided die in the morning. If he rolls a 1, then he does his laundry immediately (with zero days of delay). Otherwise, he delays for one day and repeats the experiment the following morning. Let R be the number of days a relaxed student delays laundry. What is $\operatorname{Ex}(R)$?

  Example: If the student rolls a 2 the first morning, a 5 the second morning, and a 1 the third morning, then he delays for $R = 2$ days.

  Solution. If we regard doing laundry as a failure, then the mean time to failure is $1/(1/6) = 6$. However, this counts the day laundry is done, so the number of days delay is $6 - 1 = 5$. Alternatively, we could derive the answer as follows:

  $$\operatorname{Ex}(R) = \sum_{k=0}^{\infty} \operatorname{Pr}(R > k)$$
  $$= \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \ldots$$
  $$= \frac{5}{6} \cdot \left(1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \ldots\right)$$
  $$= \frac{5}{6} \cdot \frac{1}{1 - 5/6}$$
  $$= 5$$

(c) Before doing laundry, an ***unlucky*** student must recover from illness for a number of days equal to the product of the numbers rolled on two fair, 6-sided dice. Let U be the expected number of days an unlucky student delays laundry. What is $\text{Ex}(U)$?

Example: If the rolls are 5 and 3, then the student delays for $U = 15$ days.

Solution. Let D_1 and D_2 be the two die rolls. Recall that a die roll has expectation $7/2$. Thus:

$$
\begin{aligned}
\text{Ex}(U) &= \text{Ex}(D_1 \cdot D_2) \\
&= \text{Ex}(D_1) \cdot \text{Ex}(D_2) \\
&= \frac{7}{2} \cdot \frac{7}{2} \\
&= \frac{49}{4}
\end{aligned}
$$

(d) A student is ***busy*** with probability $1/2$, ***relaxed*** with probability $1/3$, and ***unlucky*** with probability $1/6$. Let D be the number of days the student delays laundry. What is $\text{Ex}(D)$? *Leave your answer in terms of* $\text{Ex}(B)$, $\text{Ex}(R)$, *and* $\text{Ex}(U)$.

Solution.
$$
\text{Ex}(D) = \frac{1}{2}\text{Ex}(B) + \frac{1}{3}\text{Ex}(R) + \frac{1}{6}\text{Ex}(U)
$$

Problem 7. [12 points] I have twelve cards:

| 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 |

I shuffle them and deal them in a row. For example, I might get:

| 1 | 2 | 3 | 3 | 4 | 6 | 1 | 4 | 5 | 5 | 2 | 6 |

What is the expected number of adjacent pairs with the same value? In the example, there are two adjacent pairs with the same value, the 3's and the 5's.

We can award partial credit only if you show your work.

Solution. Consider an adjacent pair. The left card matches only one of the other 11 cards, which is equally likely to be in any of the 11 other positions. Therefore, the probability that an adjacent pair matches is $1/11$. Since there are 11 adjacent pairs, the expected number of matches is $11 \cdot 1/11 = 1$ by linearity of expectation.

Problem 8. [12 points] Each time a baseball player bats, he hits the ball with some proba-
bility. The table below gives the hit probability and number of chances to bat next season
for five players.

player	prob. of hit	# chances to bat
Player A	1/3	300
Player B	1/4	200
Player C	1/4	400
Player D	1/5	250
Player E	2/5	500

(a) Let X be the total number times these five players hit the ball next season. What
is $\text{Ex}(X)$?

Solution. The expected number of events that happen is equal to the sum of the
event probabilities. So we have:

$$\text{Ex}(X) = 300 \cdot \frac{1}{3} + 200 \cdot \frac{1}{4} + 400 \cdot \frac{1}{4} + 250 \cdot \frac{1}{5} + 500 \cdot \frac{2}{5}$$
$$= 100 + 50 + 100 + 50 + 200$$
$$= 500$$

(b) Give a nontrivial upper bound on $\Pr(X \geq 1500)$ and justify your answer. *Do not*
assume that hits happen mutually independently.

Solution. We use the Markov Inequality:

$$\Pr(X \geq c) \leq \frac{\text{Ex}(X)}{c}$$

This gives:

$$\Pr(X \geq 1500) \leq \frac{\text{Ex}(X)}{1500}$$
$$= \frac{1}{3}$$

(c) Using a Chernoff inequality, give a nontrivial upper bound on $\Pr(X \leq 400)$. For
this part, you *may* assume that all hits happen mutually independently.

Solution. We use the Chernoff Inequality:

$$\Pr(X \leq (1 - \delta)\text{Ex}(X)) \leq e^{-\delta^2 \text{Ex}(X)/2}$$

This gives:

$$\Pr(X \leq 400) = \Pr(X \leq (1 - 1/5)\text{Ex}(X))$$
$$\leq e^{-(1/5)^2 500/2}$$
$$= e^{-10}$$

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