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Lecture 16: Counting

1 Recall

Product rule:

$$|A_1 \times \cdots \times A_n| = |A_1| \cdot \cdots \cdot |A_n|.$$

Use when choosing from A_1 **and** from A_2 **and** etc.

Bijection rule: If $f : A \to B$ is a bijection, then $|A| = |B|$.

Sum rule: If $A_1, \ldots, A_n$ are pairwise disjoint,

$$|A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|.$$

Use when choosing from A_1 **or** A_2 **or** etc.

2 Generalized Product Rule

If A is a set of length k sequences where there are exactly

- n_1 possible first entries,
- n_2 possible second entries **no matter what first entry whas chosen**,
- n_i possible ith entries **no matter what first $i-1$ entries were chosen** (for each $1 \le i \le n$),

then $|A| = n_1 \cdot \cdots \cdot n_k$.

We saw example of a shuffled deck of cards: 52 options for the first card, 51 options remaining for the second card, 50 options remaining for the 3rd card, etc., down to 1 option remaining for the last card. Total count is $52 \cdot 51 \cdot \cdots \cdot 1 = 52!$.

Crucial observation: the **set** of possible second cards changes depending on the first choice, but the **number** of possible second cards doesn't depend on the first choice.

Note: a ***permutation*** of a set S is a sequence that contains every element of S exactly once. If S has n elements, there are $n!$ permutations.

One more example: US one dollar bills have an 8 digit serial number, and they frequently have repeated digits! Who has a dollar bill with them? Is some digit used more than once? The number of serial numbers **without** repeated digits is $10 \cdot 9 \cdot 8 \cdots 4 \cdot 3$ by the generalized product rule. Total number of serials is 10^8 by the standard product rule. Fraction that avoids repeats is $\approx .018$, or 1.8%.

Negative example: how many length 3 serials *abc* have distinct digits *and* those digits are increasing left-to-right? Let's try gen product rule. How many options for first digit? $0, 1, 2, \ldots, 7$, so 8 options. How many options for the second digit? Depends! If $a = 7$ then only one option, $b = 8$! If $a = 0$, then $b \in \{1, 2, \ldots, 8\}$! Number of options changes depending on a, so we cannot apply the generalized product rule.

3 Division Rule

So far, if $f : A \to B$ is a ***bijection***, $|A| = |B|$. This is also called a 1-to-1 correspondence, because each item on the left connects to exactly 1 item on the right.

More general is a k-to-1 correspondence: if $f : A \to B$ is such that every $a \in A$ has exactly 1 arrow out, and every $b \in B$ has exactly k arrows in, then $|B| = |A|/k$.

Example of division rule: knights of the round table! n knights must sit around the round table, but consider seatings equivalent if rotations of each other, e.g., $abcd \equiv bcda \equiv cdab \equiv dabc$. Let P be the set of perms of n knights, and C the set of cyclic orders, i.e., equivalence classes. Each cyclic order comes from precisely n permutations, so this is an n-to-1 mapping. So $|C| = |P|/n = n!/n = (n-1)!$.

Back to increasing serials! The number of len-3 increasing serials is the same as the number of size-3 subsets of $\{0, 1, \ldots, 9\}$, because we care only about which 3 digits are chosen. (We can write them in increasing order afterwards.)

So P is the set of permutations of $\{0, 1, \ldots, 9\}$, and S is the set of size-3 subsets of $\{0, 1, \ldots, 9\}$. Define $f(a_0 a_1 \cdots a_9)$ as the set formed by the first three digits, $\{a_0, a_1, a_2\}$. E.g., $f(7365128049) = \{3, 6, 7\}$.

f is a total function, so every $p \in P$ has exactly 1 arrow out. How many arrows in does each $s \in S$ have? For example, how many permutations p map to $s = \{3, 6, 7\}$? Gotta have $3, 6, 7$ in the first 3 slots (in some order), and 0124589 in the last 7 slots (in some order), so $3! \cdot 7!$. How many map to $\{0, 1, 8\}$? Still $k = 3! \cdot 7!$. This is true for any subset of 3 digits, so f is a $(3! \cdot 7!)$-to-1 correspondence. It follows that $|S| = |P|/(3! \cdot 7!) = 10!/(3! \cdot 7!)$.

Another way to think about it: choose an *ordered* sequence of 3 distinct digits, then form a set. If A is the set of ordered sequences, then $|A| = 10 \cdot 9 \cdot 8 = 10!/7!$. Then we have a map $g : A \to S$, where (a, b, c) maps to $\{a, b, c\}$. This is 3!-to-1, because each of the 3! orderings of a, b, c maps to the set $\{a, b, c\}$. So $|S| = |A|/3! = 10 \cdot 9 \cdot 8/(3 \cdot 2 \cdot 1) = 10!/(7! \cdot 3!)$.

In general, the number of ways to choose a subset of size r from a set of size n is

$$\frac{n!}{(n-r)! \cdot r!} = \binom{n}{r} = \text{“}n \text{ choose } r\text{”}.$$

This is so useful! How many ways to select 4 volunteers from a class of 350? $\binom{350}{4} = 350!/(4!\,346!)$. How many ways to select 3 pizza toppings from 15 choices? $\binom{15}{3}$. When flipping 100 coins one at a time, how many sequences end up with 50 heads and 50 tails? I.e., how many length 100 bitstrings have exactly 50 ones? This is in bijection with size-50 subsets of $\{1, 2, \ldots, 100\}$, so $\binom{100}{50}$. (This is only about 8% of all 2^{100} bitstrings!)

Note: will generalize n-choose-r in recitation, with the bookkeeper rule.

4 Counting With Sequences / Recipes

Idea: figure out how to construct your objects with a sequence of choices, then count those choices with generalized product rule. Make sure everything you construct lies in the set you're trying to count, and that everything in your set can be created in exactly one way!

Let's count Poker hands!

Standard deck of cards has 13 ranks (A,2,3,. . . ,9,10,J,Q,K) and 4 suits (clubs, hearts, spades, diamonds). One of each combination, so $13 \times 4 = 52$ cards total.

A poker hand is a subset of 5 cards.

How many hands? $\binom{52}{5}$.

4.1 4 of a kind

How many are 4-of-a-kind? Must have all 4 cards of some rank, plus one more card. Recipe: let's form sequences (R, C), where R is a rank (for the 4 of a kind) and C is any card with rank other than R (for the last card). Can think of this as a sequence of choices:

- Choose rank R for the 4-of-a-kind,
- Choose extra card C, with rank different from R.

Use (R, C) to form a hand $\{Rc, Rh, Rd, Rs, C\}$. Claim this is a bijection between $A :=$ "pairs (R, C) as described", and $B :=$ "hands that have 4-of-a-kind".

For example, $(5, Ah)$ produces the hand $\{5d, Ah, 5s, 5h, 5c\}$. Why is this a bijection?

- Does every sequence (R, C) produce a 4-of-a-kind hand?
- Does every 4-of-a-kind hand come from precisely one sequence (R, C)?

Yes, and yes! So it's a bijection, and $|B| = |A|$. Can compute $|A|$ with generalized product rule: 13 choices for the rank, and 48 choices for the extra card, so $13 \cdot 48$ is our answer.

4.2 All 4 suits

How many poker hands have all 4 suits? Can try to count with this recipe:

- rank for club
- rank for heart
- rank for spade

- rank for diamond
- one extra card

There are $13 \cdot 13 \cdot 13 \cdot 13 \cdot 48$ ways to fill out this recipe. Is this our answer?

Seems pretty convincing, right? How do we check it? What do we need to ask and verify?

Does each sequence of choices produce a hand with all 4 suits? Yes!

Does each hand with all 4 suits come from exactly one such sequence? Oops, no! $\{6d, Qh, As, 2c, Js\}$ comes from $(2, Q, A, 6, Jd)$ but also from $(2, Q, J, 6, As)$. But we're saved: it's a 2-to-1 mapping! The suit with two cards can swap the two cards, but that's it. So the answer is $13^4 \cdot 48/2$.

That's one common thing that can go wrong: not a bijection, since some hands can be made in multiple ways. Can often be salvaged with division rule, but not always.

Here's a different recipe that doesn't need the division rule:

- Choose the suit S that has two cards; let the other three suits be T_1, T_2, T_3 *in alphabetical order.*
- Choose two ranks $\{R_a, R_b\}$ for that suit
- Choose a rank R_1 to pair with T_1
- Choose a rank R_2 to pair with T_2
- Choose a rank R_3 to pair with T_3

Our hand above is *forced* to pick $S = spades$, $\{R_a, R_b\} = \{A, J\}$, $R_1 = 2$ for clubs, $R_2 = 6$ for diamonds, and $R_3 = Q$ for hearts. Never any ambiguity.

4.3 At least one pair

At least two cards with the same rank.

- Choose rank for pair, 13 choices
- Choose two suits for the pair, $\binom{4}{2}$ choices
- Choose 3 more cards $\binom{50}{3}$ choices.

E.g., $(6, \{h, d\}, \{Ac, 2c, 3d\})$ gives hand $\{6h, 6d, Ac, 2c, 3d\}$, which has a pair!

Does every hand produced in this way have a pair? Yes!

Does every hand with a pair come from this recipe exactly once? Oops, no! $\{Ah, As, 2c, 2d, 3h\}$ comes 2 different ways. $\{Ah, Ad, Ac, 3c, 5s\}$ comes in 3 different ways! $\{Ah, Ad, Ac, Qc, Qs\}$ comes in 4 different ways! $\{Ah, Ad, Ac, As, 7d\}$ comes in 6 different ways! It's not even a k-to-1 map! Oh no, we're doomed!

Better to count hands that *don't* have a pair:

- choose a subset $\{r1, r2, r3, r4, r5\}$ of five distinct ranks, $\binom{13}{5}$ ways. (Assume $r1 < r2 < \cdots < r5$.)
- Choose a suit for each rank, 4^5 ways.

The number of hands with no pair is $\binom{13}{5} \cdot 4^5$.

Alternatively, choose an *ordered* list of any 5 cards with distinct ranks: $52 \cdot 48 \cdot 44 \cdot 40 \cdot 36$ choices. Then divide by 5! because every hand comes from 5! ordered lists. So the number of hands with no pairs is $52 \cdot 48 \cdot 44 \cdot 40 \cdot 36/5!$. (Looks different, but it's the same as before!)

So the number of hands *with* at least one pair is $\binom{52}{5} - \binom{13}{5} \cdot 4^5$. (Note: about 49.3% of poker hands have at least one pair.)