6.262: Discrete Stochastic Processes 3/7/11

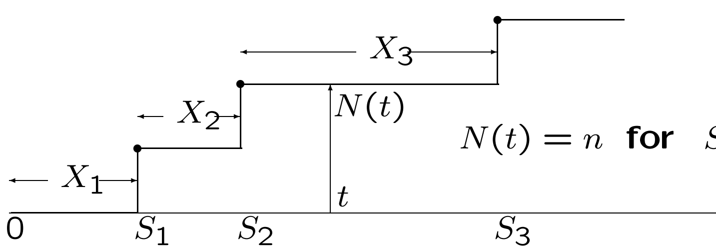
Lecture 10: Renewals and the SLLN

Outline:

- Arrival processes and renewal processes
- Convergence WP1 and the SLLN
- Proof of convergence WP1 theorem
- The strong law with a 4th moment
- SLLN and WLLN
- Strange aspects of the SLLN

Renewal processes

Recall that an arrival process can be specified by it arrival epochs, $\{S_1, S_2, \dots,\}$, or by its interarrival intervals, $\{X_1, X_2, \dots,\}$, or its counting process, $\{N(t); t > 0\}$.



$N(t) = n$ for $S_n \le t < S_{n+1}$

Def: A renewal process is an arrival process for which the interarrival intervals $X_1, X_2, \dots$ are IID.

Renewal proceesses are characterized by ‘starting over’ at each renewal.

We will see later exactly what this means, but intuititively it means that complex processes can be broken into discrete time intervals.

Renewal theory treats the gross characteristics of this (how many intervals occur per unit time, laws of large numbers about long term behavior, etc.)

The local characteristics can then be studied without worrying about long term interactions.

Example: Markov chains, for any given recurrent state, have renewals on successive visits to that state.

It is almost obvious that the intervals between visits to a given recurrent state are independent, but we look at this carefully later.

A non-obvious result that will arise from studying renewals is that the expected recurrence time between visits to state i is π_i. We could have derived that from Markov chains directly, but using renewals is cleaner.

The whole theory for Markov chains with a countably infinite state space will come from renewal processes.

Example: G/G/m queues (queues with a general IID interarrival distribution, a general IID service interval, and m servers)

The queue is assumed empty at $t < 0$ and an arrival is assumed at $t = 0$.

A very complicated interaction goes on between arrivals and departures, until finally the queue empties out.

After some interval depending on this busy period, a new arrival occurs. This can be taken as a renewal.

Proof of theorem on convergence WP1

Thm: let $\{Y_n;\ n \geq 1\}$ be rv's satisfying $\sum_{n=1}^{\infty} \mathsf{E}\left[|Y_n| < \infty\right]$. Then $\{Y_n; n \geq 1\}$ converges to 0 WP1.

That is, we want to prove that $\{\omega \in \Omega\ :\ \lim_{n\to\infty} Y_n(\omega) = 0\} = 1$.

Proof: For any $\alpha > 0$ and any finite integer $m \geq 1$, Markov says that

$$\Pr\left\{\sum_{n=1}^{m} |Y_n| > \alpha\right\} \leq \frac{\mathsf{E}\left[\sum_{n=1}^{m} |Y_n|\right]}{\alpha} = \frac{\sum_{n=1}^{m} \mathsf{E}\left[|Y_n|\right]}{\alpha} \leq \frac{\sum_{n=1}^{\infty} \mathsf{E}\left[|Y_n|\right]}{\alpha}.$$

$$\Pr\left\{\sum_{n=1}^{m} |Y_n| > \alpha\right\} \le \frac{\sum_{n=1}^{\infty} \mathsf{E}\,[|Y_n|]}{\alpha}.$$

Let $A_m = \{\omega : \sum_{n=1}^{m} |Y_n(\omega)| > \alpha\}$**, so that**

$$\Pr\left\{\sum_{n=1}^{m} |Y_n| > \alpha\right\} \;=\; \Pr\{A_m\} \;\le\; \frac{\sum_{n=1}^{\infty} \mathsf{E}\,[|Y_n|]}{\alpha}. \quad (1)$$

Since $|Y_n(\omega)| \ge 0$**, we have** $A_m \subseteq A_{m+1}$ **for** $m \ge 1$**. Thus the left side of (1), as a function of** m**, is a nondecreasing bounded sequence of real numbers. Thus**

$$\lim_{m\to\infty} \Pr\left\{\sum_{n=1}^{m} |Y_n| > \alpha\right\} \;=\; \lim_{m\to\infty} \Pr\{A_m\} \;\le\; \frac{\sum_{n=1}^{\infty} \mathsf{E}\,[|Y_n|]}{\alpha}.$$

Also, by property (9) (nesting) of the probability axioms,

$$\Pr\left\{\bigcup_{m=1}^{\infty} A_m\right\} = \lim_{m\to\infty} \Pr\{A_m\}$$

Summarizing, $A_m = \{\omega : \sum_{n=1}^{m} |Y_n(\omega)| > \alpha\}$ **and**

$$\Pr\left\{\bigcup_{m=1}^{\infty} A_m\right\} \;\le\; \frac{\sum_{n=1}^{\infty} \mathsf{E}\,[|Y_n|]}{\alpha}.$$

For given ω**,** $\sum_{n=1}^{m} |Y_n(\omega)|$ **is a sequence (in** m**) of real numbers that is nondecreasing in** m**. Either all these real numbers are upper bounded by** α**, in which case**

$$\sum_{n=1}^{\infty} |Y_n(\omega)| \le \alpha; \qquad \omega \notin \bigcup_{m=1}^{\infty} A_m$$

or $\sum_{n=1}^{m} |Y_n(\omega)| > \alpha$ **for some** m**, and**

$$\sum_{n=1}^{\infty} |Y_n(\omega)| > \alpha; \qquad \omega \in \bigcup_{m=1}^{\infty} A_m$$

Thus

$$\Pr\left\{\omega \,:\, \sum_{n=1}^{\infty} |Y_n(\omega)| > \alpha\right\} \;\le\; \frac{\sum_{n=1}^{\infty} \mathsf{E}\,[|Y_n|]}{\alpha}.$$

For $\alpha > \sum_n \mathsf{E}[|Y_n|]$

$$\Pr\left\{\omega : \sum_{n=1}^{\infty} |Y_n(\omega)| \le \alpha\right\} \;>\; 1 - \frac{\sum_{n=1}^{\infty} \mathsf{E}[|Y_n|]}{\alpha}.$$

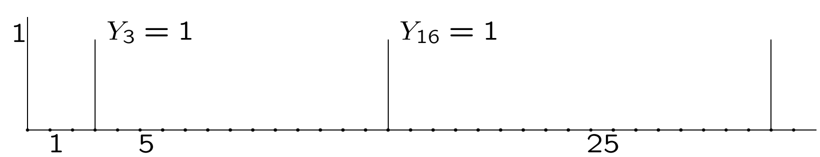
If $\sum_n |Y_n(\omega)| \le \alpha$ **for a given** ω**, then** $\sum_n |Y_n(\omega)|$ **converges and** $\lim_{m\to\infty} \sum_{n=m}^{\infty} |Y_n(\omega)| = 0$**. Thus**

$$\Pr\left\{\omega : \sum_{n=1}^{\infty} |Y_n(\omega)| \le \alpha\right\} \Longrightarrow \lim_{n\to\infty} |Y_n(\omega)| = 0$$

$$\Pr\left\{\omega : \lim_{n\to\infty} |Y_n(\omega)| = 0\right\} \;>\; 1 - \frac{\sum_{n=1}^{\infty} \mathsf{E}[|Y_n|]}{\alpha}.$$

Let $\alpha \to \infty$**. Then** $\Pr\{\omega : \lim_{n\to\infty} |Y_n(\omega)| = 0\} = 1$**.** □

Example: (convergence in probability but not WP1). $\{Y_n; n \ge 1\}$**. For each** $j \ge 0$**,** $Y_n = 1$ **for an equiprobable choice of** $n \in [5^j, 5^{j+1})$ **and** $Y_n = 0$ **otherwise.**



For every ω, j**,** $Y_n(\omega)$ **is 1 for some** $n \in [5^j, 5^{j+1})$ **and is 0 elsewhere. Thus** $Y_n(\omega)$ **does not converge for any** ω**; i.e.,** $\{Y_n;\, n \ge 1\}$ **doesn't converge WP1.**

Note that $\mathsf{E}[|Y_n|] = 1/(5^{j+1} - 5^j)$ **for** $5^j \le n < 5^{j+1}$ **and thus** $\sum_n \mathsf{E}[|Y_n|] = \infty$**, so the theorem doesn't apply.**

However, $\lim_{n\to\infty} \Pr\{|Y_n| > \epsilon\} = 0$ **for all** $\epsilon > 0$**, so** $\{Y_n;\, n \ge 1\}$ **converges in probability.**

The strong law of large numbers

Now that we have a convenient property implying convergence WP1, we use this property to prove the strong law of large numbers.

Theorem: Let $\{X_n; n \geq 1\}$ be a sequence of IID rv's satisfying $\mathsf{E}[|X|] < \infty$. For each $n \geq 1$, let $S_n = \sum_{m=1}^{\infty} X_n$. Then

$$\Pr\left\{\omega \in \Omega : \lim_{n\to\infty} \frac{S_n(\omega)}{n} = \overline{X}\right\} = 1$$

This is stated more tersely as $\Pr\left\{\lim_n S_n/n = \overline{X}\right\} = 1$, more tersely yet as $\lim_n S_n/n = \overline{X}$ WP1 and still more tersely as $S_n/n \stackrel{\text{WP1}}{\longrightarrow} \overline{X}$. The meaning of the theorem is complicated and the terse forms sometimes conceal this meaning.

Discussion: The strong law (and convergence WP1) is quite different from the other forms of convergence. It focusses directly on sample paths from $n = 1$ to ∞.

This makes it more difficult to talk about the rate of convergence as $n \to \infty$.

It is connected directly to the standard notion of convergence of a sequence of numbers applied to the sample paths. The power of this will be more apparent when looking at renewal processes.

Most of the heavy lifting with the SLLN has been done via the analysis of convergence WP1.

Proof of SLLN assuming $\overline{X} = 0$ and $\mathsf{E}\left[X^4\right] < \infty$:

$$\begin{aligned}\mathsf{E}\left[S_n^4\right] &= E\left(\sum_{i=1}^{n} X_i\right)\left(\sum_{j=1}^{n} X_j\right)\left(\sum_{k=1}^{n} X_k\right)\left(\sum_{\ell=1}^{n} X_\ell\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n} \mathsf{E}\left[X_i X_j X_k X_\ell\right],\end{aligned}$$

Consider any given term with $i = 1$ and $j, k, \ell > 1$.

Note that X_1 (the first rv of the n-tuple $(X_1, \dots, X_n)$) is independent of X_j, X_k, X_ℓ for each $j, k, \ell > 1$.

Thus $\mathsf{E}\left[X_1 X_j X_k X_\ell\right] = 0$ for terms with $j, k, \ell > 1$.

Similarly, all terms in which any one of i, j, k, ℓ is different from the rest is 0.

There are then two kinds of nonzero terms.

In the first, there are n terms (one for each of the n choices for i) in which $i = j = k = \ell$.

For the second kind of nonzero term, there are $n(n-1)$ terms for which $i = j$ and $k = \ell$. There are also $n(n-1)$ terms with $i = k$ and $j = \ell$, and $n(n-1)$ more with $i = \ell$ and $j = k$.

$$\mathsf{E}\left[S_n^4\right] = n\mathsf{E}\left[X^4\right] + 3n(n-1)(\mathsf{E}\left[X^2\right])^2$$

Now $\mathsf{E}\left[X^4\right]$ is the second moment of the rv X^2, so $(\mathsf{E}\left[X^2\right])^2 \le \mathsf{E}\left[X^4\right]$. Thus

$$\mathsf{E}\left[S_n^4\right] = [n + 3n(n-1)]\mathsf{E}\left[X^4\right] \le 3n^2\mathsf{E}\left[X^4\right]$$

$$\sum_{n=1}^{\infty} \frac{\mathsf{E}\left[S_n^4\right]}{n^4} \le 3\mathsf{E}\left[X^4\right] \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

From the theorem on convergence WP1,

$$\Pr\Big\{\omega : \lim_{n\to\infty} S_n^4(\omega)/n^4 = 0\Big\} = 1$$

For every ω **such that** $\lim_{n\to\infty} S_n^4(\omega)/n^4 = 0$**, we see that** $\lim_{n\to\infty} |S_n/n| = 0$**. Thus,**

$$\Pr\left\{\omega : \lim_{n\to\infty} \frac{S_n(\omega)}{n} = 0\right\} = 1 \qquad \square$$

The ability to go, on a sample path basis, from $\{S_n^4(\omega)/n^4;\, n \geq 1\}$ **to** $\{|S_n(\omega)/n|;\, n \geq 1\}$ **is the key to much of the usefulness of the strong law.**

Example: Consider the Bernoulli process with $\mathsf{p}_X(1) = p$**. Then** $\mathsf{E}[X] = p$ **and, according to the theorem, the set of sample paths** $\{\omega : \lim_n S_n/n = p\}$ **has probability 1.**

Consider another Bernoulli process with $\mathsf{p}_X(1) = p' \neq p$**. Now** $\Pr\{\omega : \lim_n S_n/n = p\} = 0$ **and, with no change of events,** $\Pr\{\omega : \lim_n S_n/n = p'\} = 1$**.**

There are uncountably many choices for p**, so there are uncountably many events, each of probability 1 for its own** p**.**

This partitions the sample space into an uncountable collection of events, each with probability 1 for its own p**, plus events with no convergence.**

There is nothing wrong here, but these events are peculiar and must be treated carefully.

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