

---

## Quiz

---

There are 3 questions, each with several parts. We attempted to put the easier parts of each question toward the beginning of that question.

You have 2 and a half hours to finish the quiz. If any part of any question is unclear, please ask.

The blue books are for scratch paper only. Please put your answers and an explanation of your reasoning in the white books.

Please put your name on each white booklet you turn in.

The questions have been designed to require very little computation if done in the most insightful way. Explore the question carefully and think about the simplest cases before launching into a messy solution.

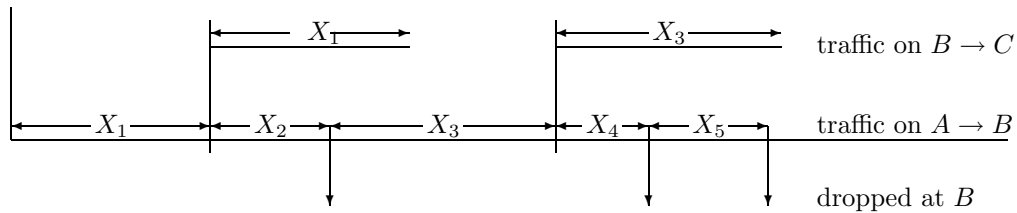
Partial credit: Careful reasoning will receive generous partial credit even if the final answer is incorrect, and correct answers with incorrect or inadequate reasons will not receive full credit.

Our attempt will be to match your grade on a question to the level of understanding, both intuitive and mathematical, that you have exhibited rather than the ability to manipulate a large number of equations.

Your explanations need not appear to be rigorous, but should be adequate as a full explanation to a fellow student, leaving nothing in doubt.

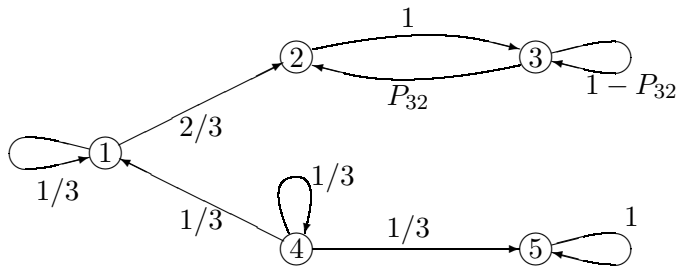
**Problem 1:** An infinite sequence of packets are waiting to be sent, one after the other, from point  $A$  to an intermediate point  $B$  and then on to  $C$ . The lengths of the packets are IID rv's and thus  $X_i$ , the time intervals required to send successive packets from  $A$  to  $B$  are IID rv's assumed here to be exponential of rate  $\lambda$ . Thus the arrival epochs of packets at point  $B$  form a Poisson process of rate  $\lambda$ .

At point  $B$ , the receiver for the  $A \rightarrow B$  link, after completely receiving a packet, passes it to the transmitter for the  $B \rightarrow C$  link to send on to point  $C$ . The designer of point  $B$  had never taken 6.262 and knew nothing about queueing. Thus, instead of providing storage for packets, any packet received on link  $A \rightarrow B$  is dropped if the transmitter for the  $B \rightarrow C$  link is currently transmitting an earlier packet. The time to send an undropped packet on the  $B \rightarrow C$  link is the same as that required for that packet on the  $A \rightarrow B$  link.



- a) Let  $D_n$  be the event that packet  $n$  is dropped. Find  $\Pr\{D_2\}$ . Hint: Express this event as a relationship between  $X_1$  and  $X_2$ .
- b) For  $n > 2$ , find the probability that packets 1 through  $n$  are all successful, *i.e.*,  $\Pr\{\bigcap_{i=1}^n D_i^c\}$ .
- c) For  $n > 2$ , find the probability that packets 2 through  $n$  are all dropped, *i.e.*,  $\Pr\{\bigcap_{i=2}^n D_i\}$ . Hint: Parts b) and c) are solved in very different ways
- d) Find  $\Pr\{D_3 | D_2\}$ ,  $\Pr\{D_3 | D_2^c\}$ , and  $\Pr\{D_3\}$ . Hint: Don't assume that these are all the same.
- e) Find the distribution of the  $k$ th idle period on the link  $B \rightarrow C$ .
- f) Does the sequence of beginning busy periods on the link  $B \rightarrow C$  constitute the sequence of renewal epochs of a renewal process?
- g) Now suppose that  $\{X_n; n \geq 1\}$  are IID and continuous but not exponential. Re-solve parts a) and b) for this case.

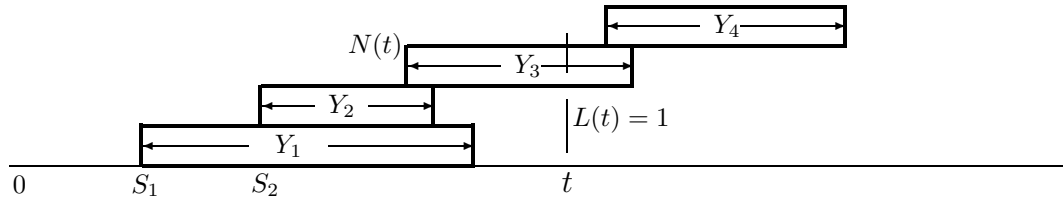
**Problem 2:** Consider the following finite-state Markov chain.



- a) Identify the transient states and identify each class of recurrent states.
- b) For each recurrent class, find the steady-state probability vector  $\boldsymbol{\pi} = (\pi_1, \dots, \pi_5)$  for that class.
- c) Find the following  $n$ -step transition probabilities,  $P_{ij}^n = \Pr\{X_n = j \mid X_0 = i\}$  as a function of  $n$ . Give a brief explanation of each (no equations are required).

- i)  $P_{44}^n$
- ii)  $P_{45}^n$
- iii)  $P_{41}^n$
- iv)  $P_{43}^n + P_{42}^n$
- v)  $\lim_{n \rightarrow \infty} P_{43}^n$

**Problem 3:** Consider a  $(G/G/\infty)$  ‘queueing’ system. That is the arriving customers form a renewal process, *i.e.*, the interarrival intervals  $\{X_n; n \geq 1\}$  are IID. You may assume throughout that  $E[X] < \infty$ . Each arrival immediately enters service; there are infinitely many servers, so one is immediately available for each arrival. The service time  $Y_i$  of the  $i$ th customer is a rv of expected value  $\bar{Y} < \infty$  and is IID with all other service times and independent of all inter-arrival times. There is no queueing in such a system, and one can easily intuit that the number in service never becomes infinite since there are always available servers.



a) Give a simple example of distributions for  $X$  and  $Y$  in which this system never becomes empty. Hint: Deterministic rv’s are fair game.

b) We want to prove Little’s theorem for this type of system, but there are no renewal instants for the entire system. As illustrated above, let  $N(t)$  be the renewal counting process for the arriving customers and let  $L(t)$  be the number of customers in the system (*i.e.*, receiving service) at time  $t$ . In distinction to our usual view of queueing systems, assume that there is no arrival at time 0 and the first arrival occurs at time  $S_1 = X_1$ . The  $n$ th arrival occurs at time  $S_n = X_1 + \dots + X_n$ .

Carefully explain why, for each sample point  $\omega$  and each time  $t > 0$ ,

$$\int_0^t L(t, \omega) dt \leq \sum_{i=1}^{N(t, \omega)} Y_i(\omega)$$

c) Find the limit as  $t \rightarrow \infty$  of  $\frac{1}{t} \sum_{i=1}^{N(t, \omega)} Y_i(\omega)$  and show that this limit exists WP1.

d) Assume that the service time distribution is bounded between 0 and some  $b > 0$ , *i.e.*, that  $F_Y(b) = 1$ . Carefully explain why

$$\int_0^{t+b} L(\tau, \omega) d\tau \geq \sum_{i=1}^{N(t, \omega)} Y_i(\omega)$$

e) Find the limit as  $t \rightarrow \infty$  of  $\frac{1}{t} \int_0^t L(\tau, \omega) d\tau$  and show that this limit exists WP1.

MIT OpenCourseWare  
<http://ocw.mit.edu>

6.262 Discrete Stochastic Processes  
Spring 2011

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.