MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Martingales II

Content.

1. Review
2. Some applications of Optional Stopping Theorem
3. Martingale Convergence Theorem

1 Review

Definition 1 (Martingale)**.** *$\{M_t\}$ is a martingale with respect to $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}$ if it satisfies:*

1. $M_t \in \mathcal{F}_t, \quad t \geq 0$
2. $\mathbb{E}|M_t| < \infty, \quad t \geq 0$
3. $\mathbb{E}[M_t|\mathcal{F}_{t-1}] = M_{t-1}, \quad t \geq 1$

In other words, $\{M_t\}$ is a martingale w.r.t. $\{X_t\}$ if:

1. $M_t = f(X_0, \ldots, X_t), \quad t \geq 0$
2. $\mathbb{E}|M_t| < \infty, \quad t \geq 0$
3. $\mathbb{E}[M_t|X_0, \ldots, X_{t-1}] = M_{t-1}, \quad t \geq 1$.

For convenience, we denote it by

$$\mathbb{E}_s = \mathbb{E}[\cdot|\mathcal{F}_s] = \mathbb{E}[\cdot|X_0, \ldots, X_s],$$

and the third condition can be written as: $\mathbb{E}_{t-1}M_t = M_{t-1}$ for any $t \geq 1$.

When we say $\{M_t\}$ is a martingale without specifying the filtration, we mean that it is a martingale with respect to its natural filtration, i.e. $\mathcal{F}_t = \sigma(M_0, \ldots, M_t)$. We consider it as a special case of the definition.

Now if $\{M_t\}$ is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(X_0, \ldots, X_t)$, it is also a martingale with respect to its filtration $\sigma(M_0, \ldots, M_t)$. In fact, by the tower property, we have:

$$\mathbb{E}[M_t|M_0, \ldots, M_{t-1}] = \mathbb{E}[\mathbb{E}[M_t|\mathcal{F}_t]|M_0, \ldots, M_{t-1}] = \mathbb{E}[M_{t-1}|M_0, \ldots, M_{t-1}] = M_{t-1}.$$

Properties of a martingale $\{M_t\}$

- $\mathbb{E}_s M_t = M_{t \wedge s}$
- $\mathbb{E}M_t = \mathbb{E}M_0, \forall t \geq 0$
- If M_t is a martingale, and τ is a stopping time, then $Y_t = M_{t\wedge\tau}$ is a martingale.
- Side note: $\mathbb{E}_n$ also works like a martingale: $\mathbb{E}_n\mathbb{E}_m = \mathbb{E}_{n\wedge m}$, $\mathbb{E}\mathbb{E}_n = \mathbb{E}_n\mathbb{E} = \mathbb{E}$. In fact, you can also define $\mathbb{E}_\tau$ and even have $\mathbb{E}_\tau M_t = M_{\tau\wedge t}$. But we won't do it in this class.

Let A_t be the gambler's ruin Markov chain starting from k. Now let's consider the simple random walk S_t starting from $S_t = k$, and $S_t = S_{t-1} + X_t$ with $\mathbb{P}(X_t = \pm 1) = \frac{1}{2}$. Let,

$$\tau = \inf\{t : S_t = 0 \text{ or } S_t = n\}.$$

Notice that $A_t = S_{t\wedge\tau}$. Since S_t is a martingale, it follows that its stopped martingale A_t is a martingale as well. This implies a good property of gambler's ruin Markov chain, which is

$$\mathbb{E}A_t = A_0 = k.$$

We will use the definitions and notations of S_t and A_t for several times in this lecture.

Theorem 1 (Optional Stopping Theorem). *If $\{M_t\}$ is a martingale and τ is a stopping time such that $\{M_t\}$ is uniformly integrable and $\mathbb{P}(\tau < \infty) = 1$, then*

$$\mathbb{E}M_\tau = \mathbb{E}M_0.$$

Proposition 1 (Uniformly integrable martingales). *The following propositions about uniformly integrable martingales (u.i.M.) hold:*

1. *$M_t = \mathbb{E}[Z|\mathcal{F}_t]$ for any Z such that $\mathbb{E}|Z| < \infty$ is always u.i.M.*
2. *If there exists $G(t)$ such that $G(t)/t \to \infty$ as $t \to \infty$. If $\sup_t \mathbb{E}[G(M_t)] < \infty$, then M_t is u.i.M.*
3. *If $|M_t - M_{t-1}| \leq c < \infty$ and $\mathbb{E}\tau < \infty$, then $Y_t = M_{t\wedge\tau}$ is u.i.M.*

$\{S_n\}$ is not uniformly integrable. Indeed, the magnitude of $|S_n|$ is approximately $O(\sqrt{n})$. For any b, one can always find some $N \approx b^2$ such that $\mathbb{E}[|S_N|1\{|S_N| \geq b\}] \geq c$, so $\sup_n \mathbb{E}[|S_n|1\{|S_n| \geq b\}] \not\to 0$ as $b \to \infty$. Hence, S_n is not uniformly integrable.

2 Some applications of O.S.T.

2.1 Gambler's Ruin

For the **gambler's ruin** problem, we start with $A_0 = k$, and we want to find $\mathbb{P}[win] = \mathbb{P}[A_\infty = n]$.

Note that $A_t = S_{t\wedge\tau}$ is a u.i.M (since A_t is bounded), therefore

$$n\mathbb{P}[win] = \mathbb{E}A_\tau = \mathbb{E}A_0 = k,$$

as

$$A_\tau = \begin{cases} 0, & \text{"ruined"} \\ n, & \text{"won"} \end{cases}.$$

Therefore, $\mathbb{P}[win] = \frac{k}{n}$.

Now let $M_t = S_t^2 - t = (S_{t-1} + X_t)^2 - t = S_{t-1}^2 + 2X_tS_{t-1} + X_t^2 - t = S_{t-1}^2 - (t-1) + 2X_tS_{t-1} = M_{t-1} + 2X_tS_{t-1}$. Therefore,

$$\mathbb{E}_{t-1}M_t = M_{t-1}.$$

$M_{t\wedge\tau}$ is uniformly integrable since the increment $|M_t - M_{t-1}| = 2|X_t S_{t-1}|$ is bounded.
Therefore, by OST, we have

$$\frac{k}{n} \cdot n^2 - \mathbb{E}\tau = \mathbb{E}[M_\tau] = M_0 = k^2,$$

and thus, $\mathbb{E}\tau = k(n-k)$.

2.2 Null recurrence of S_t

We start with $S_0 = k$. Let $\tau_1 = \inf\{t : S_t = 0\}$, and $B_t = S_{t\wedge\tau_1}$. One can think of B_t as a Markov chain with 0 the absorbing state.
We know from recurrence of S_t that $\tau_1 < \infty$ a.s.. We also know that $\mathbb{E}S_t = \mathbb{E}B_t = \mathbb{E}S_0 = k$.

If B_t were a u.i.M, then OST applies, we will have $\mathbb{E}B_{\tau_1} = k$. However, by definition, $B_{\tau_1} = 0$ a.s., so $\mathbb{E}B_{\tau_1} = 0 \neq k$. By Proposition 2(3), the only thing that prevents B_t from being a u.i.M is $\mathbb{E}\tau = \infty$. Therefore, S_t is null recurrent.

2.3 Gambler's Ruin in the asymmetric case

For the asymmetric case, i.e. $S_t = S_{t-1} + X_t$ with $\mathbb{P}(X_t = 1) = p$ and $\mathbb{P}(X_t = -1) = 1 - p$, one can use the following two martingales to compute $\mathbb{P}[win]$ and $\mathbb{E}\tau$:

1. $M_t = S_t - (2p-1)t$
2. $N_t = e^{\lambda S_t - t\psi_X(\lambda)}$, where $\psi_{X_1}(\lambda) = \ln M_{X_1}(\lambda)$

From OST, we have

$$\mathbb{E}S_\tau - (2p-1)\mathbb{E}\tau = \mathbb{E}M_\tau = M_0 = k$$

and

$$e^{\lambda n}\mathbb{P}[win] + \mathbb{P}[ruined] = \mathbb{E}N_\tau = N_0 = e^{\lambda k},$$

with some $\lambda (= \ln \frac{p}{1-p})$ such that $\psi_{X_1}(\lambda) = 0$.

3 Martingale Convergence Theorem

Think of M_t as the price of stock. At time $t-1$, you decide to move your possession of stock to F_t shares, where $F_t \in \mathcal{F}_{t-1}$ is determined by all the observed information at time $t-1$. Then the value of your portfolio at time t is

$$V_t = F_0 M_0 + F_1(M_1 - M_0) + \ldots + F_t(M_t - M_{t-1}) \triangleq \int_0^t F\, dM.$$

Proposition 2. *If M_t is a martingale, then V_t is a martingale. In particular, $\mathbb{E}V_t = \mathbb{E}V_0$.*

The important consequence is that if you start with F_0 shares priced at M_0 then no trading strategy (and no finite cash-out time) can yield an expectation different from what you had $\mathbb{E}[F_0 M_0]$ in the beginning. Assuming the market price is a martingale with respect to the same filtration $\mathcal{F}_t$ that determines the available information you have to execute the trading decisions.

Definition 2. *Starting $S_0 = 0$, define $T_k = \inf\{t \geq S_{k-1} : M_t \leq a\}$, $S_k = \inf\{t \geq T_k : M_t \geq b\}$. Define $U_n(a,b)$ =# of upcrossings of (a,b) in $0 \leq t \leq n$, i.e.*

$$U_n(a,b) = \sup\{k : S_k \leq n\}.$$

Lemma 1 (Upcrossing Lemma)**.**

$$\mathbb{E}[U_n(a,b)] \leq \frac{\mathbb{E}(M_n - a)_-}{b-a}.$$

Proof. Starting with $F_0 = 0$ and do trading: buy 1 share when $M_t \leq a$ and sell it when $M_t \geq b$. Since $V_0 = 0$, we have

$$V_n \geq (b-a)U_n + (M_n - a) \wedge 0 = (b-a)U_n - (M_n - a)_-.$$

Since V_n is a martingale, it follows from Optional Stopping Theorem that

$$\mathbb{E}U_n \leq \frac{\mathbb{E}(M_n - a)_-}{b-a}.$$

□

Theorem 2. *If M_n is a martingale such that $\mu \quad \sup_n \mathbb{E}|M_n| < \infty$, then there exists an integrable random variable M_∞ such that*

$$M_n \overset{\text{a.s.}}{\to} M_\infty\,, \quad \textit{and} \quad \mathbb{E}[|M_\infty|] \leq \mu < \infty\,.$$

If M_t is u.i.M, then $M_t \overset{L_1}{\to} M_\infty$ and

$$M_t = \mathbb{E}[M_\infty|\mathcal{F}_t].$$

Remark: Note that if M_t is u.i.M. then $\mu < \infty$ automatically. Thus, the second part of the theorem shows that every u.i.M. is in fact a Doob martingale.

Proof. **Proof of part 1**: Fix $b > a$,

$$U(a,b) = \lim_{n\to\infty} U_n(a,b).$$

By the **upcrossing lemma**, we have

$$\mathbb{E}U_n(a,b) \leq \frac{\mathbb{E}(M_n - a)_-}{b-a} \leq \sup_n \frac{\mathbb{E}|M_n| + |a|}{b-a} < \infty.$$

Therefore, by **Monotone Convergence Theorem**, we have

$$\mathbb{E}U(a,b) = \lim_{n\to\infty} \mathbb{E}U_n(a,b) \leq \sup_n \frac{\mathbb{E}|M_n| + |a|}{b-a} < \infty.$$

This imples that,

$$\mathbb{P}(U(a,b) = \infty \text{ for any } b > a, a, b \in \mathbb{Q}) = 0.$$

So with probability 1 the trajectory M_n intersects any arbitrary small interval only finitely many times. Thus there must exist a (possibly extended real-valued) random variable M_∞ such that $M_n \overset{a.s.}{\to} M_\infty$.

To show that M_∞ is in fact integrable (and hence real-valued) we use Fatou's lemma:

$$\mathbb{E}[|M_\infty|] = \mathbb{E}[\liminf_{n\to\infty} |M_n|] \leq \liminf_{n\to\infty} \mathbb{E}[|M_n|] \leq \mu < \infty\,.$$

Proof of part 2: To show $M_t = \mathbb{E}[M_\infty|\mathcal{F}_t]$, it suffices to show that for any $B \in \mathcal{F}_t$, we have

$$\mathbb{E}M_\infty 1_B = \mathbb{E}M_t 1_B.$$

For any $m \geq t$, we have

$$\mathbb{E}M_m 1_B = \mathbb{E}[\mathbb{E}_t[M_m 1_B]] = \mathbb{E}[1_B M_t].$$

Since $M_m 1_B \overset{a.s.}{\to} M_\infty 1_B$ and $\{M_m 1_B\}$ is uniformly integrable, it follows that $M_m 1_B \overset{L_1}{\to} M_\infty 1_B$. Therefore,

$$\mathbb{E}[M_t 1_B] = \lim_{m\to\infty} \mathbb{E}[M_m 1_B] = \mathbb{E}[M_\infty 1_B].$$

□

Corollary 1. *If $M_n \geq 0$, M_n is a martingale, then it converges almost surely to integrable M_∞.*

Proof. Since for any n,

$$\mathbb{E}|M_n| = \mathbb{E}M_n = \mathbb{E}M_0,$$

it follows that

$$\sup_n \mathbb{E}|M_n| < \infty.$$

□

In particular, $M_n = X_1 \dots X_n$ such that $X_n \geq 0, \mathbb{E}X_n = 1$. Then, M_n converges almost surely.

4 Further topics

Martingale and stopping time theory is rich subject. The key omissions are:

- A lot of results about martingales are also available for submartingales (i.e. when $\mathbb{E}_{t-1}[M_t] \geq M_{t-1}$) and supermartingales (i.e. when $\mathbb{E}_{t-1}[M_t] \leq M_{t-1}$).
- Maximal inequalities for martingales/submartingales/supermartingales). These establish results similar to Kolmogorov's maximal inequalities (for sums of independent r.v.s) but for general martingales. To get a flavor of such results, if $M_0 = 0$ then

$$\mathbb{P}[\max_{0\leq t\leq n} M_t > b] = \mathbb{P}[U_n(0,b) \geq 1] \leq \frac{1}{b}\mathbb{E}[|M_n|]\,,$$

where in the last step we applied the upcrossing Lemma and Markov's inequality. So in particular, in the setting of convergence theorem we see that life-time maximum of M_t is of the order of μ. Other maximal inequalities bound p-th norm of the maximum in terms of the p-th norm of M_n etc.