

## In-Class Midterm

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### Your information

Name: .....

MIT ID: .....

### Instructions

You have 80 minutes to solve the following five problems.

This is a closed-book midterm.

*Keep stapled.*

*Do not distribute outside of class.*

*You can use this page as scratch paper if you want. Scratch space will not be graded.*

*More scratch space is available at the end of the document. Additional scratch paper is available on request.*

# 1 Existence and uniqueness of Euclidean projections

Let  $\Omega \subseteq \mathbb{R}^n$  be a nonempty set.

- (a) [7pts] Give a formal argument as to why, when  $\Omega$  is *closed and convex*, the Euclidean projection of any point  $x \in \mathbb{R}^n$  exists and is unique. As a reminder, the Euclidean projection of  $x$  is a solution to the nonlinear optimization problem

$$\begin{aligned} \min_y & \|y - x\|_2 \\ \text{s.t. } & y \in \Omega. \end{aligned}$$

- (b) [7pts] If the assumption of *closedness* is removed, is the existence of a projection still guaranteed? For those cases in which a projection exists, is such a projection still guaranteed to be unique? For each, give either a proof or a counterexample.
- (c) [7pts] If the assumption of *convexity* is removed, is the existence of a projection still guaranteed? For those cases in which a projection exists, is such a projection still guaranteed to be unique? For each, give either a proof or a counterexample.

(Remember: a set is closed if “it contains its boundary”, that is, the limit of any converging sequence of points that belong to the set belongs to the set. For example, the interval  $[1, 2] := \{x \in \mathbb{R} : 1 \leq x \leq 2\}$  is a closed set, while the interval  $(1, 2] := \{x \in \mathbb{R} : 1 < x \leq 2\}$  is not. You can take for granted that the union of two closed sets is closed.)



## 2 Lipschitz continuity of projections

Let  $\Omega \subseteq \mathbb{R}^n$  be a closed convex set, and let

$$\begin{aligned}\Pi_{\Omega}(x) &:= \arg \min_y \frac{1}{2} \|y - x\|_2^2 \\ &\text{s.t. } y \in \Omega\end{aligned}$$

denote the projection of point  $x \in \mathbb{R}^n$  onto  $\Omega$ . (Recall that Euclidean projections onto a closed and convex set exist and are unique.)

[20pts] Show that  $\Pi_{\Omega}$  is 1-Lipschitz continuous, that is,

$$\|\Pi_{\Omega}(x) - \Pi_{\Omega}(x')\|_2 \leq \|x - x'\|_2 \quad \forall x, x' \in \mathbb{R}^n.$$

*(Hint: start by writing the first-order optimality conditions for  $z := \Pi_{\Omega}(x)$  and  $z' := \Pi_{\Omega}(x')$ . In particular, two vectors must have a nonpositive inner product (obtuse angle) with  $z' - z$ . What happens if you sum the inequalities and use Cauchy-Schwarz?)*



### 3 An optimization problem

Consider the optimization problem

$$\begin{aligned} \min_{x,y} \quad & \frac{1}{2}x^2 + \frac{1}{2}(y+3)^2 \\ \text{s.t.} \quad & x^2 + y^2 \geq 2 \\ & x + y \geq 0 \end{aligned}$$

- (a) [3pts] Is the objective function, seen as a function from  $\mathbb{R}^2$  to  $\mathbb{R}$ , convex? Provide a proof or a counterexample.
- (b) [3pts] Is the feasible set a convex subset of  $\mathbb{R}^2$ ? Provide a proof or a counterexample.
- (c) [8pts] Write down the KKT conditions at a generic feasible point  $(x, y)$ .
- (d) [6pts] Check whether the KKT conditions hold at points  $(x, y) = (1, -1)$  and  $(x, y) = (-1, 1)$ .



## 4 Optimality conditions for semidefinite programs

Consider the optimization problem

$$\begin{aligned} \min_X & f(X) \\ \text{s.t.} & X \succeq 0 \\ & X \in \mathbb{S}^n, \end{aligned}$$

where  $f$  is a convex and differentiable function and  $\mathbb{S}^n$  denotes the set of symmetric  $n \times n$  matrices with real coefficients.

(a) [10pts] Consider the point  $X_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

- If  $\nabla f(X_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , can you conclude that  $X_1$  is a minimizer of  $f$ ? Can you conclude that  $X_1$  is *not* the minimizer of  $f$ ?
- What about  $\nabla f(X_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ?

(b) [10pts] Consider the point  $X_2 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ .

- If  $\nabla f(X_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , can you conclude that  $X_2$  is a minimizer of  $f$ ? Can you conclude that  $X_2$  is *not* the minimizer of  $f$ ?
- What about  $\nabla f(X_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ?

For each answer, provide a (short) formal argument.

*(Hint: consider the first-order optimality conditions for conic problems. You do not need to prove the expression for the dual/polar of the positive semidefinite cone.)*



## 5 Gradient descent

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a differentiable,  $L$ -smooth function ( $L > 0$ ) attaining a minimum at a point  $x_\star \in \mathbb{R}^n$ . Consider the iterates  $\{x_t\}$  produced by the gradient descent algorithm run with constant stepsize  $\eta = 1/L$  starting from the initial point  $x_0 = 0 \in \mathbb{R}^n$ .

- (a) [10pts] Is it guaranteed that the function value is nonincreasing, that is,

$$f(x_{t+1}) \leq f(x_t) \text{ at all times } t?$$

What if  $f$  is further assumed to be convex? For each question, either give a proof or a counterexample.

- (b) [10pts] Is it guaranteed that the distance to  $x_\star$  is nonincreasing, that is,

$$\|x_{t+1} - x_\star\|_2 \leq \|x_t - x_\star\|_2 \text{ at all times } t?$$

What if  $f$  is further assumed to be convex? For each question, either give a proof or a counterexample.

(Remember: a differentiable function is  $L$ -smooth when its gradient is  $L$ -Lipschitz continuous, that is,  $\|\nabla f(x) - \nabla f(y)\|_2 \leq L \cdot \|x - y\|_2$  for all  $x, y \in \mathbb{R}^n$ .)



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