6.7220 | 15.084 — Midterm Review

Spring 2025

Midterm administrivia

Policies

- **Cheatsheet**: one page (front and back). No other written or digital materials are permitted.
- **Attendance**: only students registered for *credit* (no listeners)
- **Scope**: Lectures 1-11 (everything seen so far)
- **Collaboration**: individual midterm (no collaboration)

Goal of midterm

- The goal of the midterm is to make sure you are comfortable with the *important ideas* seen so far
- The midterm will **not** require *complex calculations*
- The midterm will **not** require *intricate proofs*
- Questions will generally require providing arguments in favor or against a statement (*i.e.*, either a proof or a counterexample)
- Much **shorter** (and easier) than problem sets

Examples of questions

- Show that a set is convex / a cone
- Show that a function is convex / stricly convex / strongly convex
- Construct a separation oracle
- Determine the normal cone at a point
- Prove the problem admits a solution
- Is the solution unique?
- Compute the solution to an optimization problem
- Write KKT conditions for a problem
- Determine whether constraint qualifications hold
- Prove separating hyperplane theorem
- ...

How to practice

- How to prepare?
  - Look back at homework
  - Look back at lecture notes, including all the examples
  - Look back at exercises in recitation. New concepts introduced in recitation (for example, subdifferentials) will **not** appear in the midterm
  - Piazza also has some posts you might find interesting
  - Several comments in grey left in the lecture notes pointing to opportunities to try to complete arguments or think about extensions
- One meta-strategy to approach theorems:
  - Where were the hypotheses used?
  - What would happen if the hypotheses were not satisfied?
  - Does the theorem have a strong geometric intuition? Can you use the intuition to fill in the formal details?
- We are here to **help**

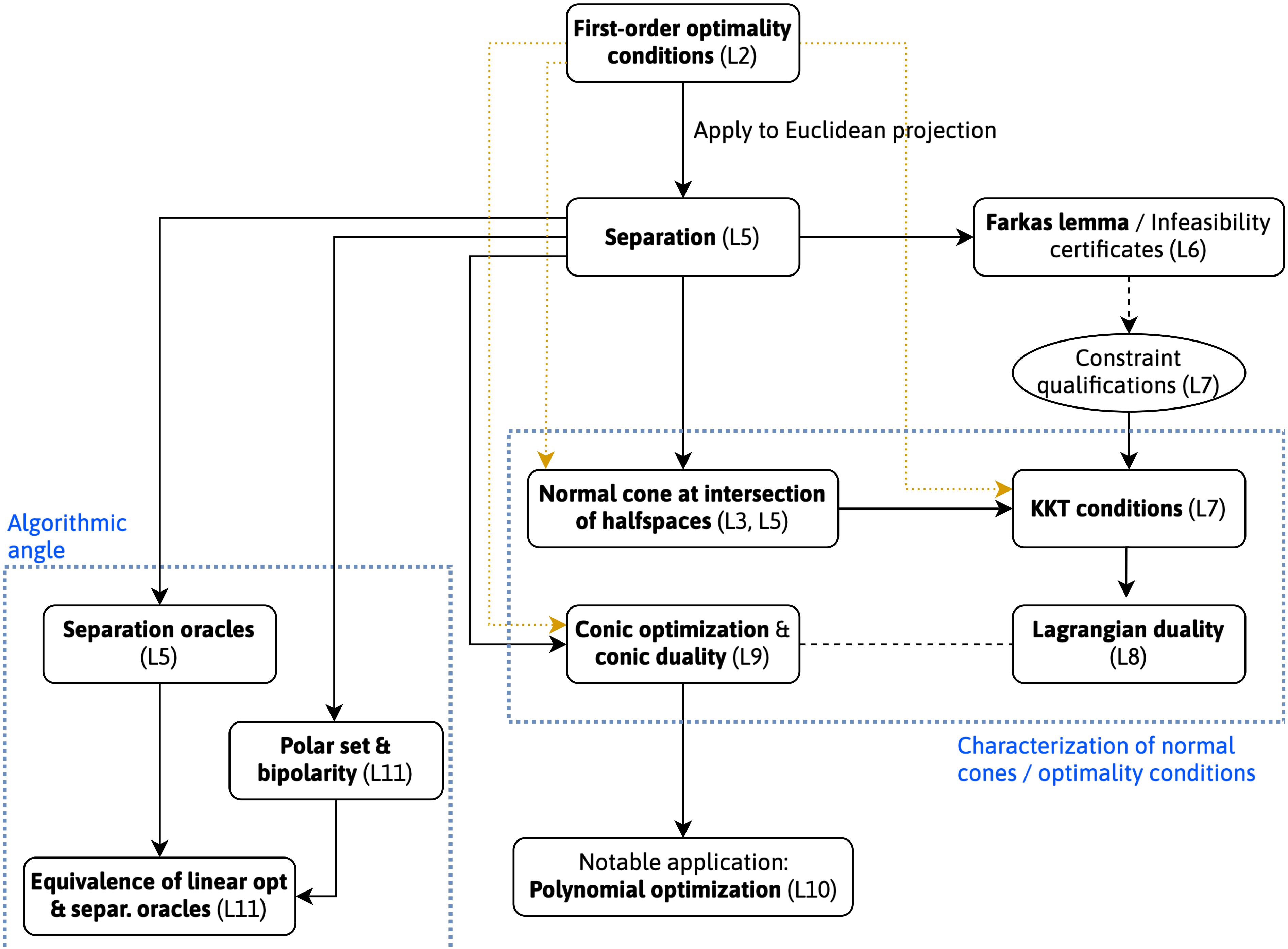
Any questions regarding midterm details?

Course so far: the big picture

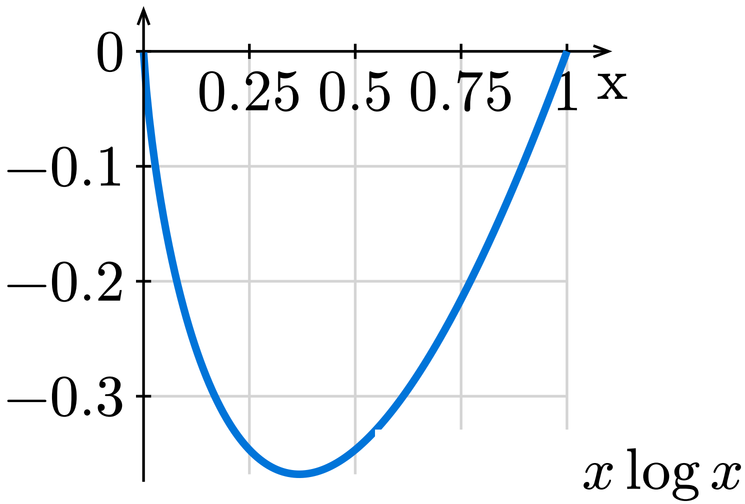
- Existence of solutions: Weierstrass theorem

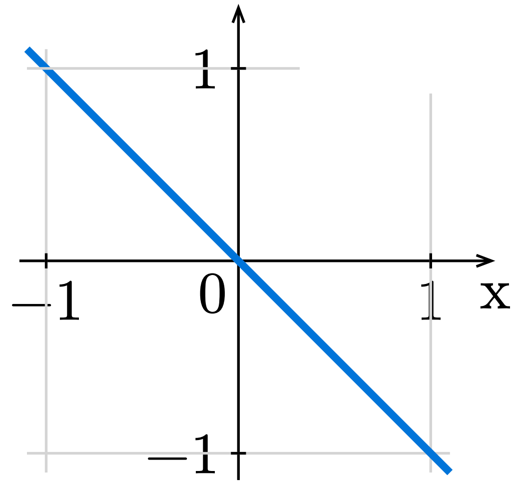
+ Three main strands:

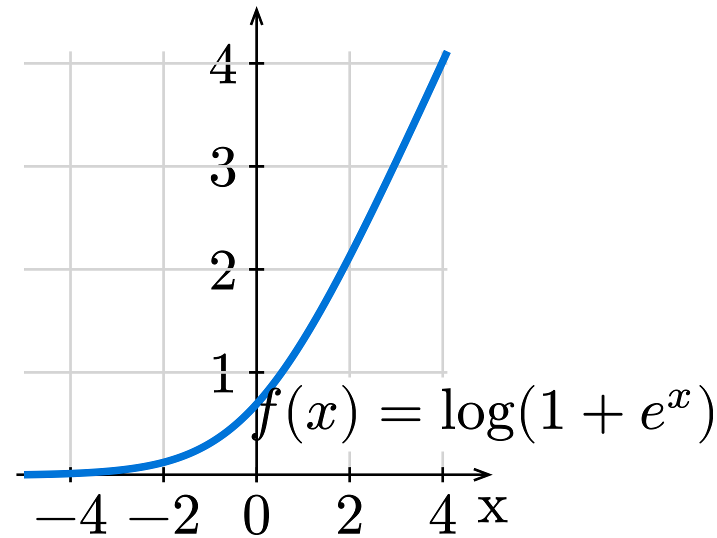
- Optimality conditions
- Separation
- Convexity


First-order optimality conditions (L2)
Apply to Euclidean projection
Separation (L5)
Farkas lemma / Infeasibility certificates (L6)
Constraint qualifications (L7)
Normal cone at intersection of halfspaces (L3, L5)
KKT conditions (L7)
Algorithmic angle
Separation oracles (L5)
Conic optimization & conic duality (L9)
Lagrangian duality (L8)
Polar set & bipolarity (L11)
Characterization of normal cones / optimality conditions
Equivalence of linear opt & separ. oracles (L11)
Notable application: Polynomial optimization (L10)

Convex functions







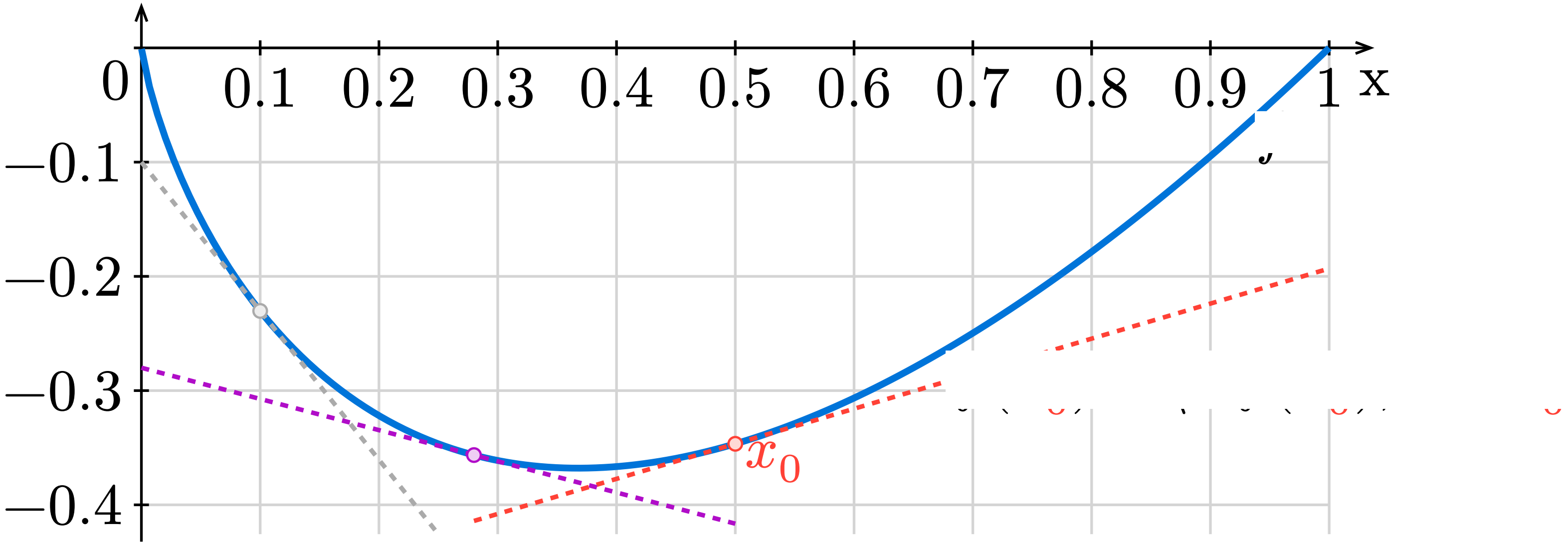
Definition

(**Convex function**) Let $\Omega \subseteq \mathbb{R}^n$ be convex.

A function $f : \Omega \to \mathbb{R}$ is *convex* if, for any two points $x, y \in \Omega$ and $t \in [0, 1]$,

$$f\big((1 - t) \cdot x + t \cdot y\big) \leq (1 - t) \cdot f(x) + t \cdot f(y).$$

- Assuming that f is not only convex but also differentiable: it lies above its linearization at any point



Theorem

Let $f : \Omega \to \mathbb{R}$ be a convex and differentiable function defined on a convex domain Ω. Then, at all $x \in \Omega$,

$$f(y) \geq \underbrace{f(x) + \langle \nabla f(x), y - x \rangle}_{\text{linearization of } f \text{ around } x} \qquad \forall y \in \Omega.$$

- Consequence: **sufficiency** of first-order optimality conditions

Theorem

Let $\Omega \subseteq \mathbb{R}^n$ be convex and $f : \Omega \to \mathbb{R}$ be a convex differentiable function. Then,

$$-\nabla f(x) \in \mathcal{N}_\Omega(x) \quad \Longleftrightarrow \quad x \text{ is a minimizer of } f \text{ on } \Omega$$

Equivalent definitions of convexity

Theorem

Let $\Omega \subseteq \mathbb{R}^n$ be a convex set, and $f : \Omega \to \mathbb{R}$ be a function. The following are equivalent definitions of convexity for f:

1. $f((1-t) \cdot x + t \cdot y) \leq (1-t) \cdot f(x) + t \cdot f(y)$ for all $x, y \in \Omega, t \in [0,1]$.
2. (If f is differentiable) $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle$ for all $x, y \in \Omega$.
3. (If f is twice differentiable and Ω is open) $\nabla^2 f(x) \succeq 0$ for all $x \in \Omega$.

Strictly and strongly convex functions

Definition

(Strict and strong convexity) Let $\Omega \subseteq \mathbb{R}^n$ be convex.

- A function $f : \Omega \to \mathbb{R}$ is *strictly convex* if, for any two distinct points $x, y \in \Omega$ and $t \in (0, 1)$,

$$f\big((1-t)\cdot x + t\cdot y\big) < (1-t)\cdot f(x) + t\cdot f(y).$$

- A function $f : \Omega \to \mathbb{R}$ is *strongly convex* with modulus $\mu > 0$ if the function

$$f(x) - \frac{\mu}{2}\|x\|_2^2$$

  is convex.

- Strong convexity implies strict convexity, and strict convexity implies convexity
- Neither of the reverse implications holds (you should be able to produce a counterexample)

Uniqueness of minimizers

- Strict convexity (and therefore strong convexity too) is useful for proving uniqueness of minimizers

Theorem

If $\Omega \subseteq \mathbb{R}^n$ is convex and f is strictly convex on Ω, then the minimizer of f on Ω (if it exists) is unique.

- Very important that Ω is convex, or the theorem is not true! (You should be able to produce a counterexample)