If we know the position x of t of an object as a function of time, we can use differentiation to calculate its velocity and its acceleration at later times.

Essentially, by taking the derivatives of the position, we know everything there is to know about the motion.

Sometimes, however, we'll want to go in the other direction.

We'll have the acceleration as a function of time and we'll want to find the velocity as a function of time, or the position as a function of time.

We'll use a technique called integration, and let's see how that works.

To begin with, suppose we have a constant acceleration.

So our acceleration a of t is some constant a 0.

In that case, we know that this constant acceleration can be written as the change in velocity delta v over some time interval delta t, and therefore that the change in velocity delta v over some time interval delta t can just be written as a 0 acceleration, the constant acceleration, times the elapsed time delta t.

And we can see this graphically if I plot acceleration versus time.

Here is my constant acceleration a 0.

Let's say this is time 0 and here is the time delta t, the change in velocity, a 0 times delta t, is just the area of this box, that box being defined by the time interval 0 to delta t, and it's the area under the curve, a 0.

So if we know the velocity at time 0 and we know the constant acceleration a 0, we can calculate the velocity at any later time in this way, because the change in velocity from that initial velocity is given by a 0 times delta t.

Now let's consider a slightly more complicated example.

Suppose the acceleration isn't constant but is changing linearly.

So for a linearly changing acceleration, I can draw this graphically as well.

Here's acceleration.

This is time equals 0.

Suppose the acceleration is increasing linearly, and I'll call this time delta t.

This is a of t.

Note that we can define an average acceleration over the interval from 0 to delta t.

The average acceleration is the change in velocity over the elapsed time.

That average acceleration will look something like that.

And so then the change in velocity, delta v, is equal to the average acceleration times the elapsed time.

And this time, notice from the diagram this product is actually the area of this trapezoid that's basically the area under the a of t function going from 0 to delta t.

We've calculated it sort of as a rectangular area involving the average acceleration, but that's also equal to the area under the time changing acceleration function.

So this is, again, the area under a of t over the time interval delta t.

Now let's consider a more generally changing acceleration as a function of time.

So let me plot this.

So suppose we have just some general function-- I'll draw it like this-- which is our acceleration as a function of time.

The change in velocity will still be the area under the curve a of t over the time interval we're interested in.

So let's suppose we want to go from this time here-- I'll call it t sub a-- and my ending time will be t sub b.

So we're interested in figuring out what the area under this curve is over the interval from t sub a to t sub b.

We can estimate this by breaking up the interval into a bunch of little rectangle.

Suppose I break this up into n rectangles just under this curve.

And so this is 1, 2, 3, 4, et cetera, up to n.

And with each one, because this is a curved graph, I can't get the rectangle to fit exactly so a little bit of it will stick out at the top of each one.

I'm not going to go through that for all of these, but you get the idea.

I'll fit these rectangles as well as I can.

The narrower I make the rectangles, the more easily I can fit them under the curve.

For each strip, for each rectangular strip, let's say from over box 1 from here to here, the change in velocity is equal to the average velocity over that time interval times the width of that rectangle, delta t.

So that's the area of the strip.

Essentially what we're saying is that if we make the rectangles narrow enough, we can treat the acceleration, the curve of the acceleration, as roughly constant over that interval.

And we can make n as big as we need to to make those rectangles very, very narrow.

So in that case, we can estimate the total change in velocity from time t sub a to time t sub b by adding up the area of all of these rectangular strips.

So in that case, we write that.

The change in velocity-- so I'll say v sub b minus v sub a-- that's the change of velocity going from time a to time b-- is equal to the sum, as j goes from 1 to n, of the area of each of these strips.

And so we want the acceleration at the i-th rectangle, so that's a of t sub j times the width of the strip.

Each strip has the same width.

We'll just call that delta t.

And that's the area of n strips.

Now for a finite-- I really should write this as approximately equal to-- because for a finite number of rectangular strips, this is just an approximation, because as I mentioned, the rectangles don't exactly fit under the curve.

They can't because the rectangle doesn't have a curved top, but the function is curved.

But the narrower I make the rectangles, or equivalently, the larger I make n, the better the approximation will be.

So what I want to do is go to the limit of an infinite number of rectangles, or equivalently, the limit of infinitesimally narrow strips.

I want delta t to go to 0.

So to make this exact, what I would write is that the change in velocity from time a to time b is equal to the limit of the sum from j equals 1 to n of the acceleration at time t sub j subject times delta t.

And we want to evaluate that limit as delta t goes to 0, or equivalently, as n, the number of rectangles, goes to infinity.

Now this is a very important expression, and we have a special way of writing it.

We can also write it as an integral.

So we write it as the integral of the function a of t dt evaluated from time equals t sub a to time equals t sub b.

And this is the area under the a of t curve, the exact area-- not the approximate area, but the exact area under the a of t curve over the interval from t of a to t of b.

This limiting sum that we've written this way on the right-hand side is called the definite integral of a of t.

And it's related to the process of integration that you've learned about in calculus, which is the inverse of taking the derivative, the inverse of differentiation.

I want to take a moment to summarize the basic principles of integration from calculus.

Let's begin by considering a function g of x with some derivative.

So consider a function g of x.

And let's assume it's a well-behaved function, by which I mean that it's continuous and differentiable over the interval that we're interested in.

So consider g of x with a derivative.

So dg dx equal to another function, which I'll call f of x.

Now note that if I add a constant to g of x, I'll still get the same derivative.

So note that the derivative with respect to x of g of x plus a constant is still equal to the same function f of x.

And this is because of the derivative of a constant is equal to 0.

Now, suppose I want to invert this process.

Then I can write that the antiderivative of f of x, the antiderivative of f of x, which I'll write as the integral of f of x, dx, is equal to g of x plus a constant.

The left-hand side of this we call an indefinite integral.

And so we see that if the derivative of g of x is f of x plus a concert, the antiderivative of f of x is g of x plus a constant.

And that can be any arbitrary constant.

Now in calculus, one learns how to calculate the indefinite integral of various functions, polynomials, trigonometric functions, logarithmic functions, et cetera.

Calculus also shows us how to compute the definite integral.

So the definite integral, the integral of f of x, dx, evaluated from x equals a to x equals b.

So this is the definite integral computed over some interval that is equal to the antiderivative at x equals b minus the antiderivative evaluated at x equals a.

And this turns out to be the area under the curve f of x in the interval between x equals a and x equals b.

Now notice that there is no arbitrary constant in the definite integral.

In the indefinite integral, we have an arbitrary constant, but in the definite integral, that arbitrary constant is determined by setting the integration limits.

So there's no arbitrary constant.

We just have this difference.

And so just to see this graphically, if I plot my function f of x and suppose this is x equals a and this is x equals b, this definite integral represents the area under the curve f of x in the interval from x equals a to x equals b.

So calculus tells us how to solve this area problem, how to compute a definite integral, from the antiderivative that you get from indefinite integration.

And so this same technique tells us how to determine the velocity from the acceleration, since we saw that that was equivalent to an area under the curve problem.

So to come back to the motion of objects, we've shown that the change in velocity of an object can be written as the definite integral of the acceleration.

So just to write that a little bit more formally first with a plot, if this is my acceleration as a function of time, we know that the time derivative of the velocity is equal to the acceleration as a function of time.

I can rewrite that as the differential dv is equal to a of t times the differential dt.

And so then I can integrate both sides of this equation by writing the integral over dv is equal to the integral of the acceleration of a of t dt.

And I can go from time equals some initial time t 0 to time equal to some later time t sub 1 on the right-hand side to make a definite integral.

And then on the left-hand side the corresponding limits are the velocity at time 0-- I'll call that v 0-- and the velocity at time 1, which I'll call v1.

So just to be clear, I'm assuming here that v 0 is equal to the velocity at time t 0, and v1 is equal to the velocity at time t sub 1.

So this is the integral of a constant over an interval of v.

This is an interval of the acceleration over the time interval in t.

And so the left-hand side-- this is just v1 minus v 0.

And the right-hand side, without specifying a of t, I can't actually evaluate this integral.

I can't specify what the antiderivative of a of t is unless I tell you what the function a of t is.

So we'll just have to leave it in terms of an integral.

And so that's just the integral of a of t dt from t equals t 0 to t equals t sub 1.

So again, this shows us that the change in the velocity from time t 0 to a later time t1 is equal to the definite integral of a of t over that integral.

I can rewrite this in terms of what the velocity is at some later time, t1, by writing the velocity at t1 is equal to v 0 plus the integral of a of t dt from t equals t 0 to t equals t1.

Note that T1 is just any later time after time t 0, where we have the initial velocity.

So a more convenient way of writing this function might be to write the velocity as a function of some later time t.

So suppose I were to do that.

I'll just rewrite this equation replacing t1 with an arbitrary time t.

So I have that v of t is equal to v 0 plus the integral of a of t dt from t equals t0 to t equals t.

Now there is something funny here, because I have t in the integration variable here.

But I have t as one of my limits here as well, whereas if I look at this expression here, there's a difference between the t in the integration variable and the limit t1.

They actually represent different things.

So to keep track of that, the notation that we generally use in physics is to call this integration variable t prime and so we write this acceleration of t prime, dt prime, with the time t prime going from t 0 to some later time t.

Now one has to be cautious here.

In some fields that prime on a variable is used to denote a derivative, a differentiation.

That's not what it means here.

I'm writing t prime just to distinguish it from the specific later time t that I want to calculate the velocity at.

And it's worth thinking about what this expression means.

This equation is identical to this earlier equation that we derived except for a change in notation.

And so let's think about what that notation means.

t prime here is the integration variable.

It's a placeholder.

Remember, the integral here, the definite integral, represents an infinite sum, an infinite sum of rectangles between a time t 0 and a later time t.

And that variable t prime-- actually, what I should do now is I should call this t prime.

This variable t prime is taking every value from t 0 to t.

So t prime is representing the running time variable for all of our strips that we're adding up over this definite integral.

So it's a placeholder variable.

We sometimes call it a dummy variable.

It's just a placeholder to represent time, whereas t, the t in the limit here without the prime, represents a specific choice of later time, some later time t where we want to calculate the velocity.

So we know the velocity at some initial time t 0.

We'd like to know the velocity at some specific later time t.

And to compute that, we have to integrate over all times running from t 0 to t.

And that running integration variable we represent as t prime just to distinguish it from the specific time t that we are trying to compute the velocity for.

So now in just the same way that we've obtained the velocity by integrating the acceleration, we can integrate again.

We can integrate the velocity to calculate the position.

So given v of t, we can show that the position at time t is equal to the position at time t 0 plus the integral of the velocity as a function of time-- I'll write this as t prime-- dt prime going from t prime equals t 0 to t prime equals t.

So this is exactly analogous to how we computed velocity from acceleration.

By integrating a second time, we can go from velocity to position.

So once we know the acceleration a of t, we can use integration to compute the velocity v of t if we know the velocity at some initial time t 0.

And we can also compute the position x of t if we know the initial position at time t 0, x 0.

So we see that given the acceleration, we can recover the velocity and the position.

And as it happens, from Newton's Second Law, if we know the forces acting on an object, that gives us the ability to compute what the acceleration is.

And then given the acceleration, we can use integration to find the velocity and the position.