How do we solve problems involving massive pulleys using Newton's laws?

As a simple example, let's look at this problem, consisting of a block of mass m1 hanging from a massive pulley that has a moment of inertia I and radius r.

We'll find the acceleration of the block, a1, and the angular acceleration of the massive pulley, alpha.

It is critical to remember that the first step in solving these problems is to define the positive x and y directions and the positive direction of rotation.

In two-dimensional problems, you're free to find any direction to be positive.

Here, we'll set our positive x and positive y like this.

Once we pick x and y, the direction of the positive rotation is now also defined by the right-hand rule.

You'll see shortly that defining positive directions at the beginning will save you from a negative sign nightmare later on.

Now let's break down Newton's laws for the different parts of the system.

First, for the block, we will write down the linear version of Newton's second law.

For the sum of forces, we have gravity pointing down and tension pointing up.

According to the convention we just defined, T1 is positive and m1g is negative.

Notice that since the block is accelerating, T1 minus m1g is not 0 but is equal to m1a1.

Notice that here we've set the signs for tension and weight, but we don't yet know which were the block is accelerating.

So we just start by writing m1a1.

And if, in the end, we get that a1 is negative, we'll know that it's actually accelerating in the negative direction.

Now let's look at the pulley.

Newton's second law in its rotational form is tau is equal to I alpha.

For a normal pulley, the string is always tangential to the side of the pulley.

So r and f are perpendicular, so that means the torque is just T1 times r.

What's the sign of this term?

Well, according to the positive direction that we defined, this torque is positive.

Again, we'll leave the sign of the I alpha term to be positive.

And alpha will turn out to be either positive or negative.

Finally, we need to connect these two equations.

Because the block is connected to the pulley using an ideal, taut, inextensible rope, a1 is going to be related to alpha.

We just need to figure out exactly how they're related.

In other words, we need to write down the constraint condition.

Let's say that the block hypothetically is moving upwards.

In this case, what's happening to the pulley?

It must be spinning clockwise to pull the rope up as the block goes in, so it has a clockwise angular velocity, which, according to our convention, is negative.

In fact, if the pulley rotates a full turn in the clockwise negative direction, the angle changes by negative 2 pi, and 2 pi times r of the rope will be pulled up.

So we can write that delta y is equal to negative r delta theta or, taking some derivatives, a1 is equal to negative r times alpha.

After setting the sign conventions, writing Newton's laws for different parts of the system, and then writing down the constraint condition, we have three equations and three unknowns for which we can now solve.