Chapter 10 Momentum, System of Particles, and Conservation of Momentum

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Law II: The change of motion is proportional to the motive force impressed, and is made in the direction of the right line in which that force is impressed.

If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force is impressed altogether and at once or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both. [1]

Isaac Newton *Principia*

10.1 Introduction

When we apply a force to an object, through pushing, pulling, hitting or otherwise, we are applying that force over a discrete interval of time, Δt. During this time interval, the applied force may be constant, or it may vary in magnitude or direction. Forces may also be applied continuously without interruption, such as the gravitational interaction between the earth and the moon. In this chapter we will investigate the relationship between forces and the time during which they are applied, and in the process learn about the quantity of momentum, the principle of conservation of momentum, and its use in solving a new set of problems involving systems of particles.

10.2 Momentum (Quantity of Motion) and Average Impulse

Consider a point-like object (particle) of mass m that is moving with velocity $\vec{\mathbf{v}}$ with respect to some fixed reference frame. The quantity of motion or the **momentum**, $\vec{\mathbf{p}}$, of the object is defined to be the product of the mass and velocity

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}} . \qquad (10.2.1)$$

Momentum is a reference frame dependent vector quantity, with direction and magnitude. The direction of momentum is the same as the direction of the velocity. The magnitude of the momentum is the product of the mass and the instantaneous speed.

Units: In the SI system of units, momentum has units of $[\mathrm{kg \cdot m \cdot s^{-1}}]$. There is no special name for this combination of units.

[1] Isaac Newton. *Mathematical Principles of Natural Philosophy.* Translated by Andrew Motte (1729). Revised by Florian Cajori. Berkeley: University of California Press, 1934. p. 13.

During a time interval Δt, a non-uniform force $\vec{\mathbf{F}}$ is applied to the particle. Because we are assuming that the mass of the point-like object does not change, Newton's Second Law is then

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} = m\frac{d\vec{\mathbf{v}}}{dt} = \frac{d(m\vec{\mathbf{v}})}{dt} . \tag{10.2.2}$$

Because we are assuming that the mass of the point-like object does not change, the Second Law can be written as

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} . \tag{10.2.3}$$

The **impulse** of a force acting on a particle during a time interval $[t, t+\Delta t]$ is defined as the definite integral of the force from t to $t+\Delta t$,

$$\vec{\mathbf{I}} = \int_{t'=t}^{t'=t+\Delta t} \vec{\mathbf{F}}(t')\,dt' . \tag{10.2.4}$$

The SI units of impulse are $[\mathrm{N}\cdot\mathrm{m}] = [\mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{s}^{-1}]$ which are the same units as the units of momentum.

Apply Newton's Second Law in Eq. (10.2.4) yielding

$$\vec{\mathbf{I}} = \int_{t'=t}^{t'=t+\Delta t} \vec{\mathbf{F}}(t')\,dt' = \int_{t'=t}^{t'=t+\Delta t} \frac{d\vec{\mathbf{p}}}{dt'}\,dt' = \int_{\vec{\mathbf{p}}'=\vec{\mathbf{p}}(t)}^{\vec{\mathbf{p}}'=\vec{\mathbf{p}}(t+\Delta t)} d\vec{\mathbf{p}}' = \vec{\mathbf{p}}(t+\Delta t) - \vec{\mathbf{p}}(t) = \Delta\vec{\mathbf{p}} . \tag{10.2.5}$$

Eq. (10.2.5) represents the integral version of Newton's Second Law: the impulse applied by a force during the time interval $[t, t+\Delta t]$ is equal to the change in momentum of the particle during that time interval.

The average value of that force during the time interval Δt is given by the integral expression

$$\vec{\mathbf{F}}_{\text{ave}} = \frac{1}{\Delta t}\int_{t'=t}^{t'=t+\Delta t} \vec{\mathbf{F}}(t')\,dt' . \tag{10.2.6}$$

The product of the average force acting on an object and the time interval over which it is applied is called the ***average impulse***,

$$\vec{\mathbf{I}}_{\text{ave}} = \vec{\mathbf{F}}_{\text{ave}}\,\Delta t . \tag{10.2.7}$$

Multiply each side of Eq. (10.2.6) by Δt resulting in the statement that the average impulse applied to a particle during the time interval $[t, t+\Delta t]$ is equal to the change in momentum of the particle during that time interval,

$$\vec{\mathbf{I}}_{\text{ave}} = \Delta \vec{\mathbf{p}}. \tag{10.2.8}$$

Example 10.1 Impulse for a Non-Constant Force

$\Delta t = 1.0\ \text{s}$

Suppose you push an object for a time $b\Delta t / 2$ in the $+x$-direction. For the first half of

$F(t)t$

the interval, you push with a force that increases linearly with time according to

$$\vec{\mathbf{F}}(t) = bt\,\hat{\mathbf{i}}, \quad 0 \le t \le 0.5\text{s} \ \text{ with } \ b = 2.0 \times 10^{1}\ \text{N} \cdot \text{s}^{-1}. \tag{10.2.9}$$

Then for the second half of the interval, you push with a linearly decreasing force,

$$\vec{\mathbf{F}}(t) = (d - bt)\hat{\mathbf{i}}, \quad 0.5\text{s} \le t \le 1.0\text{s} \ \text{ with } d = 2.0 \times 10^{1}\ \text{N} \tag{10.2.10}$$

The force vs. time graph is shown in Figure 10.3. What is the impulse applied to the object?

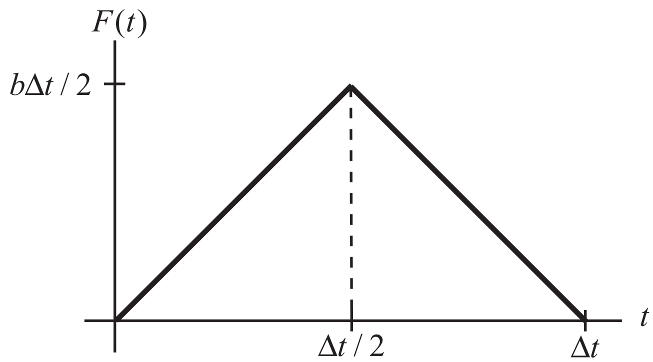

Figure 10.3 Graph of force vs. time

Solution: We can find the impulse by calculating the area under the force vs. time curve. Since the force vs. time graph consists of two triangles, the area under the curve is easy to calculate and is given by

$$\begin{aligned}\vec{\mathbf{I}} &= \left[\frac{1}{2}(b\Delta t/2)(\Delta t/2) + \frac{1}{2}(b\Delta t/2)(\Delta t/2)\right]\hat{\mathbf{i}} \\ &= \frac{1}{4}b(\Delta t)^2\,\hat{\mathbf{i}} = \frac{1}{4}(2.0\times 10^{1}\ \text{N}\cdot\text{s}^{-1})(1.0\text{s})^2\,\hat{\mathbf{i}} = (5.0\,\text{N}\cdot\text{s})\hat{\mathbf{i}}.\end{aligned} \tag{10.2.11}$$

10.3 External and Internal Forces and the Change in Momentum of a System

So far we have restricted ourselves to considering how the momentum of an object changes under the action of a force. For example, if we analyze in detail the forces acting on the cart rolling down the inclined plane (Figure 10.4), we determine that there are three forces acting on the cart: the force $\vec{\mathbf{F}}_{\text{spring, cart}}$ the spring applies to the cart; the gravitational interaction $\vec{\mathbf{F}}_{\text{earth, cart}}$ between the cart and the earth; and the contact force $\vec{\mathbf{F}}_{\text{plane, cart}}$ between the inclined plane and the cart. If we define the cart as our *system*, then everything else acts as the *surroundings*. We illustrate this division of system and surroundings in Figure 10.4.

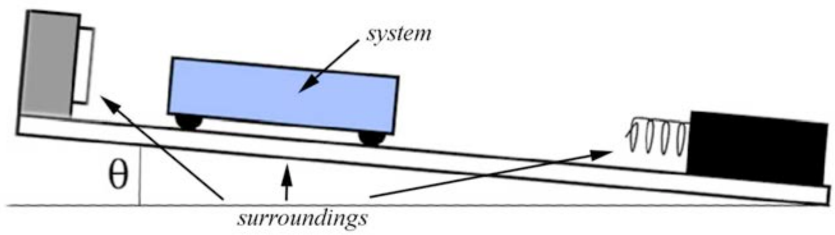

Figure 10.4 A diagram of a cart as a system and its surroundings

The forces acting on the cart are *external* forces. We refer to the vector sum of these external forces that are applied to the system (the cart) as the external force,

$$\vec{\mathbf{F}}^{\text{ext}} = \vec{\mathbf{F}}_{\text{spring, cart}} + \vec{\mathbf{F}}_{\text{earth, cart}} + \vec{\mathbf{F}}_{\text{plane, cart}} . \tag{10.3.1}$$

Then Newton's Second Law applied to the cart, in terms of impulse, is

$$\Delta\vec{\mathbf{p}}_{\text{sys}} = \int_{t_0}^{t_f} \vec{\mathbf{F}}^{\text{ext}}\, dt \equiv \vec{\mathbf{I}}_{\text{sys}} . \tag{10.3.2}$$

Let's extend our system to two interacting objects, for example the cart and the spring. The forces between the spring and cart are now *internal* forces. Both objects, the cart and the spring, experience these internal forces, which by Newton's Third Law are equal in magnitude and applied in opposite directions. So when we sum up the internal forces for the whole system, they cancel. Thus the sum of all the internal forces is always zero,

$$\vec{\mathbf{F}}^{\text{int}} = \vec{\mathbf{0}}. \tag{10.3.3}$$

External forces are still acting on our system; the gravitational force, the contact force between the inclined plane and the cart, and also a new external force, the force between the spring and the force sensor. The force acting on the system is the sum of the internal and the external forces. However, as we have shown, the internal forces cancel, so we have that

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}^{\text{ext}} + \vec{\mathbf{F}}^{\text{int}} = \vec{\mathbf{F}}^{\text{ext}}. \qquad (10.3.4)$$

10.4 System of Particles

Suppose we have a system of N particles labeled by the index $i = 1, 2, 3, \cdots, N$. The force on the i^{th} particle is

$$\vec{\mathbf{F}}_i = \vec{\mathbf{F}}_i^{\text{ext}} + \sum_{j=1,\, j\neq i}^{j=N} \vec{\mathbf{F}}_{i,j}. \qquad (10.4.1)$$

In this expression $\vec{\mathbf{F}}_{j,i}$ is the force on the i^{th} particle due to the interaction between the i^{th} and j^{th} particles. We sum over all *j* particles with $j \neq i$ since a particle cannot exert a force on itself (equivalently, we could define $\vec{\mathbf{F}}_{i,i} = \vec{\mathbf{0}}$), yielding the internal force acting on the i^{th} particle,

$$\vec{\mathbf{F}}_i^{\text{int}} = \sum_{j=1,\, j\neq i}^{j=N} \vec{\mathbf{F}}_{j,i}. \qquad (10.4.2)$$

The force acting on the system is the sum over all i particles of the force acting on each particle,

$$\vec{\mathbf{F}} = \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i = \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i^{\text{ext}} + \sum_{i=1}^{i=N} \sum_{j=1,\, j\neq i}^{j=N} \vec{\mathbf{F}}_{j,i} = \vec{\mathbf{F}}^{\text{ext}}. \qquad (10.4.3)$$

Note that the double sum vanishes,

$$\sum_{i=1}^{i=N} \sum_{j=1,\, j\neq i}^{j=N} \vec{\mathbf{F}}_{j,i} = \vec{\mathbf{0}}, \qquad (10.4.4)$$

because all internal forces cancel in pairs,

$$\vec{\mathbf{F}}_{j,i} + \vec{\mathbf{F}}_{i,j} = \vec{\mathbf{0}}. \qquad (10.4.5)$$

The force on the i^{th} particle is equal to the rate of change in momentum of the i^{th} particle,

$$\vec{\mathbf{F}}_i = \frac{d\vec{\mathbf{p}}_i}{dt}. \qquad (10.4.6)$$

When can now substitute Equation (10.4.6) into Equation (10.4.3) and determine that that the external force is equal to the sum over all particles of the momentum change of each particle,

$$\vec{\mathbf{F}}^{\text{ext}} = \sum_{i=1}^{i=N} \frac{d\vec{\mathbf{p}}_i}{dt}. \tag{10.4.7}$$

The momentum of the system is given by the sum

$$\vec{\mathbf{p}}_{\text{sys}} = \sum_{i=1}^{i=N} \vec{\mathbf{p}}_i ; \tag{10.4.8}$$

momenta add as vectors. We conclude that the external force causes the momentum of the system to change, and we thus restate and generalize Newton's Second Law for a system of objects as

$$\vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt}. \tag{10.4.9}$$

In terms of impulse, this becomes the statement

$$\Delta\vec{\mathbf{p}}_{\text{sys}} = \int_{t_0}^{t_f} \vec{\mathbf{F}}^{\text{ext}}\, dt \equiv \vec{\mathbf{I}}. \tag{10.4.10}$$

10.5 Center of Mass

Consider two point-like particles with masses m_1 and m_2. Choose a coordinate system with a choice of origin such that body 1 has position $\vec{\mathbf{r}}_1$ and body 2 has position $\vec{\mathbf{r}}_2$ (Figure 10.5).

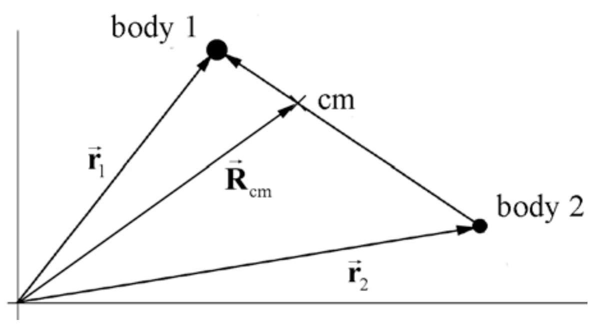

Figure 10.5 Center of mass coordinate system.

The center of mass vector, $\vec{\mathbf{R}}_{\text{cm}}$, of the two-body system is defined as

$$\vec{\mathbf{R}}_{\text{cm}} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2}. \tag{10.5.1}$$

We shall now extend the concept of the center of mass to more general systems. Suppose we have a system of N particles labeled by the index $i = 1, 2, 3, \cdots, N$. Choose a coordinate system and denote the position of the i^{th} particle as $\vec{\mathbf{r}}_i$. The mass of the system is given by the sum

$$m_{\text{sys}} = \sum_{i=1}^{i=N} m_i \tag{10.5.2}$$

and the position of the center of mass of the system of particles is given by

$$\vec{\mathbf{R}}_{\text{cm}} = \frac{1}{m_{\text{sys}}} \sum_{i=1}^{i=N} m_i \, \vec{\mathbf{r}}_i \,. \tag{10.5.3}$$

(For a continuous rigid body, each point-like particle has mass dm and is located at the position $\vec{\mathbf{r}}'$. The center of mass is then defined as an integral over the body,

$$\vec{\mathbf{R}}_{\text{cm}} = \frac{\int_{\text{body}} dm\,\vec{\mathbf{r}}'}{\int_{\text{body}} dm} \,. \tag{10.5.4}$$

Example 10.2 Center of Mass of the Earth-Moon System

The mean distance from the center of the earth to the center of the moon is $r_{em} = 3.84 \times 10^{8}\ \text{m}$. The mass of the earth is $m_e = 5.98 \times 10^{24}\ \text{kg}$ and the mass of the moon is $m_m = 7.34 \times 10^{22}\ \text{kg}$. The mean radius of the earth is $r_e = 6.37 \times 10^{6}\ \text{m}$. The mean radius of the moon is $r_m = 1.74 \times 10^{6}\ \text{m}$. Where is the location of the center of mass of the earth-moon system? Is it inside the earth's radius or outside?

Solution: The center of mass of the earth-moon system is defined to be

$$\vec{\mathbf{R}}_{cm} = \frac{1}{m_{\text{sys}}} \sum_{i=1}^{i=N} m_i \vec{\mathbf{r}}_i = \frac{1}{m_e + m_m} (m_e \vec{\mathbf{r}}_e + m_m \vec{\mathbf{r}}_m) \,. \tag{10.5.5}$$

Choose an origin at the center of the earth and a unit vector $\hat{\mathbf{i}}$ pointing towards the moon, then $\vec{\mathbf{r}}_e = \vec{\mathbf{0}}$. The center of mass of the earth-moon system is then

$$\vec{\mathbf{R}}_{cm} = \frac{1}{m_e + m_m} (m_e \vec{\mathbf{r}}_e + m_m \vec{\mathbf{r}}_m) = \frac{m_m \vec{\mathbf{r}}_{em}}{m_e + m_m} = \frac{m_m r_{em}}{m_e + m_m} \hat{\mathbf{i}} \tag{10.5.6}$$

$$\vec{\mathbf{R}}_{cm} = \frac{(7.34 \times 10^{22}\ \text{kg})(3.84 \times 10^{8}\ \text{m})}{(5.98 \times 10^{24}\ \text{kg} + 7.34 \times 10^{22}\ \text{kg})} \hat{\mathbf{i}} = 4.66 \times 10^{6}\ \text{m}\ \hat{\mathbf{i}} \tag{10.5.7}$$

The earth's mean radius is $r_e = 6.37 \times 10^6\ \text{m}$ so the center of mass of the earth-moon system lies within the earth.

Example 10.3 Center of Mass of a Rod

A thin rod has length L and mass M.

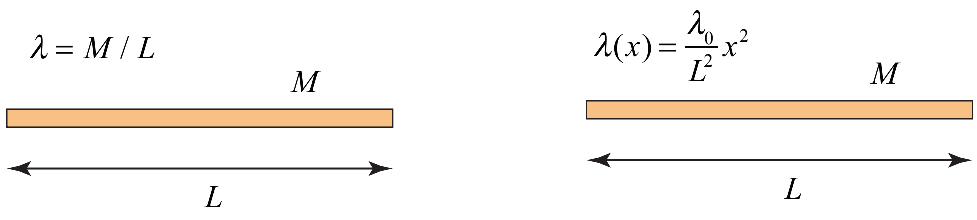

Figure 10.6 a) Uniform rod and b) non-uniform rod

(a) Suppose the rod is uniform (Figure 10.6a). Find the position of the center of mass with respect to the left end of the rod. (b) Now suppose the rod is not uniform (Figure 10.6b) with a linear mass density that varies with the distance x from the left end according to

$$\lambda(x) = \frac{\lambda_0}{L^2} x^2 \tag{10.5.8}$$

where λ_0 is a constant and has SI units $[\text{kg} \cdot \text{m}^{-1}]$. Find λ_0 and the position of the center of mass with respect to the left end of the rod.

Solution: (a) Choose a coordinate system with the rod aligned along the x-axis and the origin located at the left end of the rod. The center of mass of the rod can be found using the definition given in Eq. (10.5.4). In that expression dm is an infinitesimal mass element and $\vec{\mathbf{r}}$ is the vector from the origin to the mass element dm (Figure 10.6c).

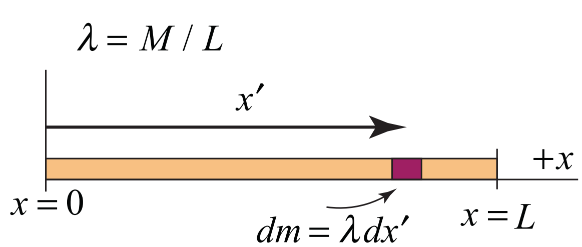

Figure 10.6c Infinitesimal mass element for rod

Choose an infinitesimal mass element dm located a distance x' from the origin. In this problem x' will be the integration variable. Let the length of the mass element be dx'. Then

$$dm = \lambda dx' \tag{10.5.9}$$

The vector $\vec{\mathbf{r}} = x'\,\hat{\mathbf{i}}$. The center of mass is found by integration

$$\vec{\mathbf{R}}_{cm} = \frac{1}{M}\int_{\text{body}} \vec{\mathbf{r}}\, dm = \frac{1}{L}\int_{x'=0}^{x} x' dx'\,\hat{\mathbf{i}} = \frac{1}{2L} x'^2\Big|_{x'=0}^{x'=L}\,\hat{\mathbf{i}} = \frac{1}{2L}(L^2 - 0)\,\hat{\mathbf{i}} = \frac{L}{2}\,\hat{\mathbf{i}}. \quad (10.5.10)$$

(b) For a non-uniform rod (Figure 10.6d),

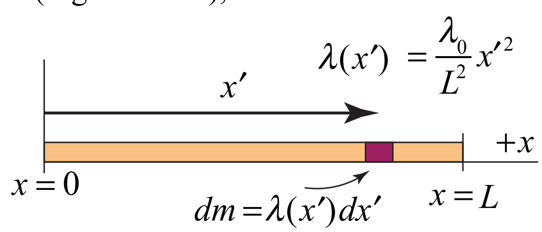

Figure 10.6d Non-uniform rod

the mass element is found using Eq. (10.5.8)

$$dm = \lambda(x')dx' = \lambda = \frac{\lambda_0}{L^2} x'^2 dx'. \quad (10.5.11)$$

The vector $\vec{\mathbf{r}} = x'\,\hat{\mathbf{i}}$. The mass is found by integrating the mass element over the length of the rod

$$M = \int_{\text{body}} dm = \int_{x'=0}^{x=L} \lambda(x')dx' = \frac{\lambda_0}{L^2}\int_{x'=0}^{x=L} x'^2 dx' = \frac{\lambda_0}{3L^2} x'^3\Big|_{x'=0}^{x'=L} = \frac{\lambda_0}{3L^2}(L^3 - 0) = \frac{\lambda_0}{3}L. \quad (10.5.12)$$

Therefore

$$\lambda_0 = \frac{3M}{L} \quad (10.5.13)$$

The center of mass is again found by integration

$$\begin{aligned} \vec{\mathbf{R}}_{cm} &= \frac{1}{M}\int_{\text{body}} \vec{\mathbf{r}}\, dm = \frac{3}{\lambda_0 L}\int_{x'=0}^{x} \lambda(x') x'\, dx'\,\hat{\mathbf{i}} = \frac{3}{L^3}\int_{x'=0}^{x} x'^3\, dx'\,\hat{\mathbf{i}} \\ \vec{\mathbf{R}}_{cm} &= \frac{3}{4L^3} x'^4\Big|_{x'=0}^{x'=L}\,\hat{\mathbf{i}} = \frac{3}{4L^3}(L^4 - 0)\,\hat{\mathbf{i}} = \frac{3}{4}L\,\hat{\mathbf{i}}. \end{aligned} \quad (10.5.14)$$

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10.6 Translational Motion of the Center of Mass

The velocity of the center of mass is found by differentiation,

$$\vec{\mathbf{V}}_{\text{cm}} = \frac{1}{m_{\text{sys}}}\sum_{i=1}^{i=N} m_i \vec{\mathbf{v}}_i = \frac{\vec{\mathbf{p}}_{\text{sys}}}{m_{\text{sys}}}. \tag{10.6.1}$$

The momentum is then expressed in terms of the velocity of the center of mass by

$$\vec{\mathbf{p}}_{\text{sys}} = m_{\text{sys}}\vec{\mathbf{V}}_{\text{cm}}. \tag{10.6.2}$$

We have already determined that the external force is equal to the change of the momentum of the system (Equation (10.4.9)). If we now substitute Equation (10.6.2) into Equation (10.4.9), and continue with our assumption of constant masses m_i, we have that

$$\vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt} = m_{\text{sys}}\frac{d\vec{\mathbf{V}}_{\text{cm}}}{dt} = m_{\text{sys}}\vec{\mathbf{A}}_{\text{cm}}, \tag{10.6.3}$$

where $\vec{\mathbf{A}}_{\text{cm}}$, the derivative with respect to time of $\vec{\mathbf{V}}_{\text{cm}}$, is the acceleration of the center of mass. From Equation (10.6.3) we can conclude that in considering the linear motion of the center of mass, the sum of the external forces may be regarded as acting at the center of mass.

Example 10.4 Forces on a Baseball Bat

Suppose you push a baseball bat lying on a nearly frictionless table at the center of mass, position 2, with a force $\vec{\mathbf{F}}$ (Figure 10.7). Will the acceleration of the center of mass be *greater than, equal to, or less than* if you push the bat with the same force at either end, positions 1 and 3

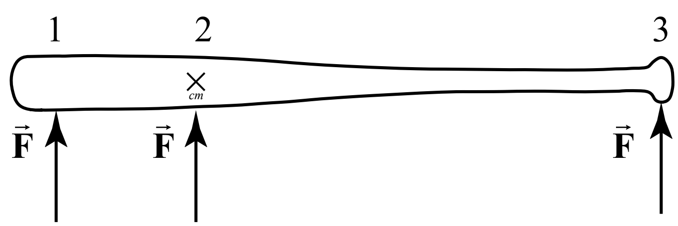

Figure 10.7 Forces acting on a baseball bat

Solution: The acceleration of the center of mass will be equal in the three cases. From our previous discussion, (Equation (10.6.3)), the acceleration of the center of mass is independent of where the force is applied. However, the bat undergoes a very different motion if we apply the force at one end or at the center of mass. When we apply the force at the center of mass all the particles in the baseball bat will undergo linear motion (Figure 10.7a).

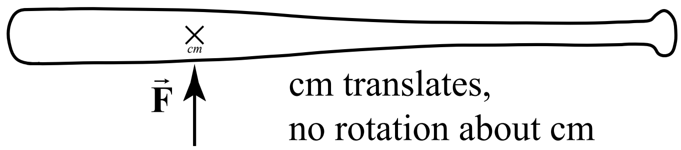

Figure 10.7a Force applied at center of mass

When we push the bat at one end, the particles that make up the baseball bat will no longer undergo a linear motion even though the center of mass undergoes linear motion. In fact, each particle will rotate about the center of mass of the bat while the center of mass of the bat accelerates in the direction of the applied force (Figure 10.7b).

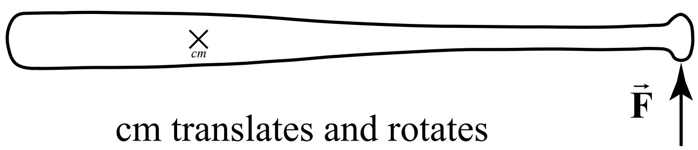

Figure 10.7b Force applied at end of bat

10.7 Constancy of Momentum and Isolated Systems

Suppose we now completely isolate our system from the surroundings. When the external force acting on the system is zero,

$$\vec{\mathbf{F}}^{\text{ext}} = \vec{\mathbf{0}}\,. \tag{10.7.1}$$

the system is called an ***isolated system***. For an isolated system, the change in the momentum of the system is zero,

$$\Delta\vec{\mathbf{p}}_{\text{sys}} = \vec{\mathbf{0}} \quad \text{(isolated system)}, \tag{10.7.2}$$

therefore the momentum of the isolated system is constant. The initial momentum of our system is the sum of the initial momentum of the individual particles,

$$\vec{\mathbf{p}}_{\text{sys},i} = m_1\vec{\mathbf{v}}_{1,i} + m_2\vec{\mathbf{v}}_{2,i} + \cdots. \tag{10.7.3}$$

The final momentum is the sum of the final momentum of the individual particles,

$$\vec{\mathbf{p}}_{\text{sys},f} = m_1\vec{\mathbf{v}}_{1,f} + m_2\vec{\mathbf{v}}_{2,f} + \cdots. \tag{10.7.4}$$

Note that the right-hand-sides of Equations. (10.7.3) and (10.7.4) are vector sums.

When the external force on a system is zero, then the initial momentum of the system equals the final momentum of the system,

$$\vec{\mathbf{p}}_{\text{sys},i} = \vec{\mathbf{p}}_{\text{sys},f} . \tag{10.7.5}$$

10.8 Momentum Changes and Non-isolated Systems

Suppose the external force acting on the system is not zero,

$$\vec{\mathbf{F}}^{\text{ext}} \neq \vec{\mathbf{0}}. \tag{10.8.1}$$

and hence the system is not isolated. By Newton's Third Law, the sum of the force on the surroundings is equal in magnitude but opposite in direction to the external force acting on the system,

$$\vec{\mathbf{F}}^{\text{sur}} = -\vec{\mathbf{F}}^{\text{ext}} . \tag{10.8.2}$$

It's important to note that in Equation (10.8.2), all internal forces in the surroundings sum to zero. Thus the sum of the external force acting on the system and the force acting on the surroundings is zero,

$$\vec{\mathbf{F}}^{\text{sur}} + \vec{\mathbf{F}}^{\text{ext}} = \vec{\mathbf{0}} . \tag{10.8.3}$$

We have already found (Equation (10.4.9)) that the external force $\vec{\mathbf{F}}^{\text{ext}}$ acting on a system is equal to the rate of change of the momentum of the system. Similarly, the force on the surrounding is equal to the rate of change of the momentum of the surroundings. Therefore the momentum of both the system and surroundings is always conserved.

For a system and all of the surroundings that undergo any change of state, the change in the momentum of the system and its surroundings is zero,

$$\Delta\vec{\mathbf{p}}_{\text{sys}} + \Delta\vec{\mathbf{p}}_{\text{sur}} = \vec{\mathbf{0}}. \tag{10.8.4}$$

Equation (10.8.4) is referred to as the ***Principle of Conservation of Momentum.***

10.9 Worked Examples

10.9.1 Problem Solving Strategies

When solving problems involving changing momentum in a system, we shall employ our general problem solving strategy involving four basic steps:

1. Understand – get a conceptual grasp of the problem.
2. Devise a Plan - set up a procedure to obtain the desired solution.
3. Carry our your plan – solve the problem!

4. Look Back – check your solution and method of solution.

We shall develop a set of guiding ideas for the first two steps.

1. Understand – get a conceptual grasp of the problem

The first question you should ask is whether or not momentum is constant in some system that is changing its state after undergoing an interaction. First you must identify the objects that compose the system and how they are changing their state due to the interaction. As a guide, try to determine which objects change their momentum in the course of interaction. You must keep track of the momentum of these objects before and after any interaction. Second, momentum is a vector quantity so the question of whether momentum is constant or not must be answered in each relevant direction. In order to determine this, there are two important considerations. You should identify any external forces acting on the system. Remember that a non-zero external force will cause the momentum of the system to change, (Equation (10.4.9) above),

$$\vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt}. \tag{10.9.1}$$

Equation (10.9.1) is a vector equation; if the external force in some direction is zero, then the change of momentum in that direction is zero. In some cases, external forces may act but the time interval during which the interaction takes place is so small that the impulse is small in magnitude compared to the momentum and might be negligible. Recall that the average external impulse changes the momentum of the system

$$\vec{\mathbf{I}} = \vec{\mathbf{F}}^{\text{ext}} \Delta t_{\text{int}} = \Delta \vec{\mathbf{p}}_{\text{sys}}. \tag{10.9.2}$$

If the interaction time is small enough, the momentum of the system is constant, $\Delta\vec{\mathbf{p}} \to \vec{\mathbf{0}}$. If the momentum is not constant then you must apply either Equation (10.9.1) or Equation (10.9.2). If the momentum of the system is constant, then you can apply Equation (10.7.5),

$$\vec{\mathbf{p}}_{\text{sys},\,i} = \vec{\mathbf{p}}_{\text{sys},\,f}. \tag{10.9.3}$$

If there is no net external force in some direction, for example the x-direction, the component of momentum is constant in that direction, and you must apply

$$p_{\text{sys},\,x,i} = p_{\text{sys},\,x,f} \tag{10.9.4}$$

2. Devise a Plan - set up a procedure to obtain the desired solution

Draw diagrams of all the elements of your system for the two states immediately before and after the system changes its state. Choose symbols to identify each mass and velocity in the system. Identify a set of positive directions and unit vectors for each state. Choose your symbols to correspond to the state and motion (this facilitates an easy interpretation, for example $(v_{x,i})_1$ represents the x-component of the velocity of object 1 in the initial state and $(v_{x,f})_1$ represents the x-component of the velocity of object 1 in the final state). Decide whether you are using components or magnitudes for your velocity symbols. Since momentum is a vector quantity, identify the initial and final vector components of the momentum. We shall refer to these diagrams as ***momentum flow diagrams***. Based on your model you can now write expressions for the initial and final momentum of your system. As an example in which two objects are moving only in the x-direction, the initial x-component of the momentum is

$$p_{\text{sys},x,i} = m_1(v_{x,i})_1 + m_2(v_{x,i})_2 + \cdots. \tag{10.9.5}$$

The final x-component of the momentum is

$$p_{\text{sys},x,f} = m_1(v_{x,f})_1 + m_2(v_{x,f})_2 + \cdots. \tag{10.9.6}$$

If the x-component of the momentum is constant then

$$p_{\text{sys},x,i} = p_{\text{sys},x,f}\,. \tag{10.9.7}$$

We can now substitute Equations (10.9.5) and (10.9.6) into Equation (10.9.7), yielding

$$m_1(v_{x,i})_1 + m_2(v_{x,i})_2 + \cdots = m_1(v_{x,f})_1 + m_2(v_{x,f})_2 + \cdots. \tag{10.9.8}$$

Equation (10.9.8) can now be used for any further analysis required by a particular problem. For example, you may have enough information to calculate the final velocities of the objects after the interaction. If so then carry out your plan and check your solution, especially dimensions or units and any relevant vector directions.

Example 10.5 Exploding Projectile

An instrument-carrying projectile of mass m_1 accidentally explodes at the top of its trajectory. The horizontal distance between launch point and the explosion is x_i. The projectile breaks into two pieces that fly apart horizontally. The larger piece, m_3, has three times the mass of the smaller piece, m_2. To the surprise of the scientist in charge, the smaller piece returns to earth at the launching station. Neglect air resistance and effects due to the earth's curvature. How far away, $x_{3,f}$, from the original launching point does the larger piece land?

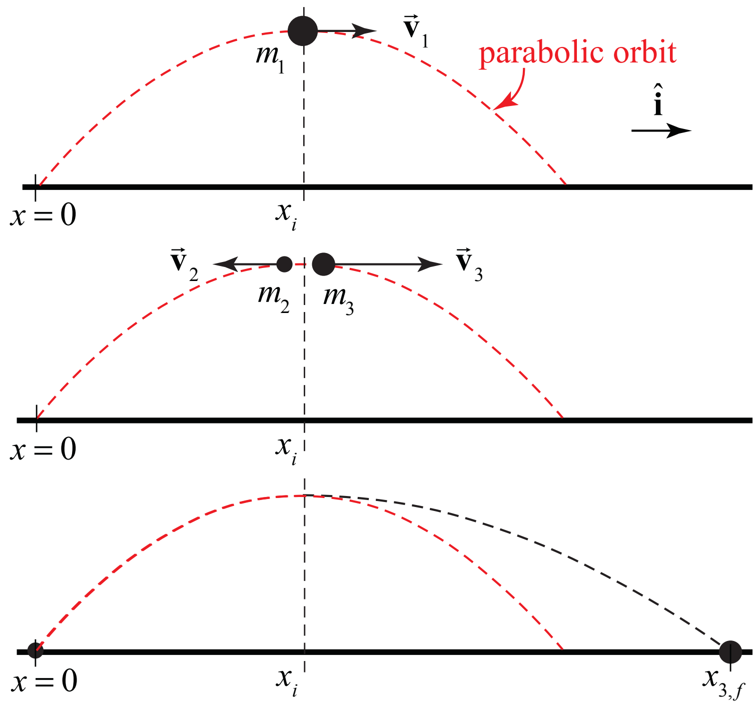

Figure 10.8 Exploding projectile trajectories

Solution: We can solve this problem two different ways. The easiest approach is utilizes the fact that the external force is the gravitational force and therefore the center of mass of the system follows a parabolic trajectory. From the information given in the problem $m_2 = m_1/4$ and $m_3 = 3m_1/4$. Thus when the two objects return to the ground the center of mass of the system has traveled a distance $R_{cm} = 2x_i$. We now use the definition of center of mass to find where the object with the greater mass hits the ground. Choose an origin at the starting point. The center of mass of the system is given by

$$\vec{\mathbf{R}}_{cm} = \frac{m_2\vec{\mathbf{r}}_2 + m_3\vec{\mathbf{r}}_3}{m_2 + m_3} \ .$$

So when the objects hit the ground $\vec{\mathbf{R}}_{cm} = 2x_i\,\hat{\mathbf{i}}$, the object with the smaller mass returns to the origin, $\vec{\mathbf{r}}_2 = \vec{\mathbf{0}}$, and the position vector of the other object is $\vec{\mathbf{r}}_3 = x_{3,f}\,\hat{\mathbf{i}}$. So using the definition of the center of mass,

$$2x_i\,\hat{\mathbf{i}} = \frac{(3m_1/4)x_{3,f}\,\hat{\mathbf{i}}}{m_1/4 + 3m_1/4} = \frac{(3m_1/4)x_{3,f}\,\hat{\mathbf{i}}}{m_1} = \frac{3}{4}x_{3,f}\,\hat{\mathbf{i}} \ .$$

Therefore

$$x_{3,f} = \frac{8}{3}x_i \, .$$

Note that the neither the vertical height above ground nor the gravitational acceleration g entered into our solution.

Alternatively, we can use conservation of momentum and kinematics to find the distance traveled. Because the smaller piece returns to the starting point after the collision, the velocity of the smaller piece immediately after the explosion is equal to the negative of the velocity of original object immediately before the explosion. Because the collision is instantaneous, the horizontal component of the momentum is constant during the collision. We can use this to determine the speed of the larger piece after the collision. The larger piece takes the same amount of time to return to the ground as the projectile originally takes to reach the top of the flight. We can therefore determine how far the larger piece traveled horizontally.

We begin by identifying various states in the problem.

Initial state, time $t_0 = 0$: the projectile is launched.

State 1 time t_1: the projectile is at the top of its flight trajectory immediately before the explosion. The mass is m_1 and the velocity of the projectile is $\vec{\mathbf{v}}_1 = v_1\hat{\mathbf{i}}$.

State 2 time t_2: immediately after the explosion, the projectile has broken into two pieces, one of mass m_2 moving backwards (in the negative x-direction) with velocity $\vec{\mathbf{v}}_2 = -\vec{\mathbf{v}}_1$. The other piece of mass m_3 is moving in the positive x-direction with velocity $\vec{\mathbf{v}}_3 = v_3\hat{\mathbf{i}}$, (Figure 10.8).

State 3: the two pieces strike the ground at time $t_f = 2t_1$, one at the original launch site and the other at a distance $x_{3,f}$ from the launch site, as indicated in Figure 10.8. The pieces take the same amount of time to reach the ground $\Delta t = t_1$ because both pieces are falling from the same height as the original piece reached at time t_1, and each has no component of velocity in the vertical direction immediately after the explosion. The momentum flow diagram with state 1 as the initial state and state 2 as the final state are shown in the upper two diagrams in Figure 10.8.

The initial momentum at time t_1 immediately before the explosion is

$$\vec{\mathbf{p}}^{sys}(t_1) = m_1\vec{\mathbf{v}}_1\,. \qquad (10.9.9)$$

The momentum at time t_2 immediately after the explosion is

$$\vec{\mathbf{p}}^{sys}(t_2) = m_2\,\vec{\mathbf{v}}_2 + m_3\,\vec{\mathbf{v}}_3 = -\frac{1}{4}m_1\vec{\mathbf{v}}_1 + \frac{3}{4}m_1\vec{\mathbf{v}}_3 \qquad (10.9.10)$$

During the duration of the instantaneous explosion, impulse due to the external gravitational force may be neglected and therefore the momentum of the system is constant. In the horizontal direction, we have that

$$m_1\,\vec{\mathbf{v}}_1 = -\frac{1}{4}m_1\vec{\mathbf{v}}_1 + \frac{3}{4}m_1\vec{\mathbf{v}}_3\,. \qquad (10.9.11)$$

Equation (10.9.11) can now be solved for the velocity of the larger piece immediately after the collision,

$$\vec{\mathbf{v}}_3 = \frac{5}{3}\vec{\mathbf{v}}_1\,. \qquad (10.9.12)$$

The larger piece travels a distance

$$x_{3,f} = v_3\,t_1 = \frac{5}{3}v_1\,t_1 = \frac{5}{3}x_i\,. \qquad (10.9.13)$$

Therefore the total distance the larger piece traveled from the launching station is

$$x_f = x_i + \frac{5}{3}x_i = \frac{8}{3}x_i\,, \qquad (10.9.14)$$

in agreement with our previous approach.

Example 10.6 Landing Plane and Sandbag

Figure 10.9 Plane and sandbag

A light plane of mass $1000\ \text{kg}$ makes an emergency landing on a short runway. With its engine off, it lands on the runway at a speed of $40\ \text{m}\cdot\text{s}^{-1}$. A hook on the plane snags a cable attached to a $120\ \text{kg}$ sandbag and drags the sandbag along. If the coefficient of friction between the sandbag and the runway is $\mu_k = 0.4$, and if the plane's brakes give an additional retarding force of magnitude $1400\ \text{N}$, how far does the plane go before it comes to a stop?

Solution: We shall assume that when the plane snags the sandbag, the collision is instantaneous so the momentum in the horizontal direction remains constant,

$$p_{x,i} = p_{x,1}. \tag{10.9.15}$$

We then know the speed of the plane and the sandbag immediately after the collision. After the collision, there are two external forces acting on the system of the plane and sandbag, the friction between the sandbag and the ground and the braking force of the runway on the plane. So we can use the Newton's Second Law to determine the acceleration and then one-dimensional kinematics to find the distance the plane traveled since we can determine the change in kinetic energy.

The momentum of the plane immediately before the collision is

$$\vec{\mathbf{p}}_i = m_p v_{p,i}\,\hat{\mathbf{i}} \tag{10.9.16}$$

The momentum of the plane and sandbag immediately after the collision is

$$\vec{\mathbf{p}}_1 = (m_p + m_s) v_{p,1}\,\hat{\mathbf{i}} \tag{10.9.17}$$

Because the x- component of the momentum is constant, we can substitute Eqs. (10.9.16) and (10.9.17) into Eq. (10.9.15) yielding

$$m_p v_{p,i} = (m_p + m_s) v_{p,1}. \tag{10.9.18}$$

The speed of the plane and sandbag immediately after the collision is

$$v_{p,1} = \frac{m_p v_{p,i}}{m_p + m_s} \tag{10.9.19}$$

The forces acting on the system consisting of the plane and the sandbag are the normal force on the sandbag,

$$\vec{\mathbf{N}}_{g,s} = N_{g,s}\,\hat{\mathbf{j}}, \tag{10.9.20}$$

the frictional force between the sandbag and the ground

$$\vec{\mathbf{f}}_k = -f_k\,\hat{\mathbf{i}} = -\mu_k N_{g,s}\,\hat{\mathbf{i}}, \tag{10.9.21}$$

the braking force on the plane

$$\vec{\mathbf{F}}_{g,p} = -F_{g,p}\,\hat{\mathbf{i}}, \tag{10.9.22}$$

and the gravitational force on the system,

$$(m_p + m_s)\vec{\mathbf{g}} = -(m_p + m_s)g\hat{\mathbf{j}}. \qquad (10.9.23)$$

Newton's Second Law in the $\hat{\mathbf{i}}$-direction becomes

$$-F_{g,p} - f_k = (m_p + m_s)a_x. \qquad (10.9.24)$$

If we just look at the vertical forces on the sandbag alone then Newton's Second Law in the $\hat{\mathbf{j}}$-direction becomes

$$N - m_s g = 0.$$

The frictional force on the sandbag is then

$$\vec{\mathbf{f}}_k = -\mu_k N_{g,s}\hat{\mathbf{i}} = -\mu_k m_s g\hat{\mathbf{i}}. \qquad (10.9.25)$$

Newton's Second Law in the $\hat{\mathbf{i}}$-direction becomes

$$-F_{g,p} - \mu_k m_s g = (m_p + m_s)a_x.$$

The x-component of the acceleration of the plane and the sand bag is then

$$a_x = \frac{-F_{g,p} - \mu_k m_s g}{m_p + m_s} \qquad (10.9.26)$$

We choose our origin at the location of the plane immediately after the collision, $x_p(0) = 0$. Set $t = 0$ immediately after the collision. The x-component of the velocity of the plane immediately after the collision is $v_{x,0} = v_{p,1}$. Set $t = t_f$ when the plane just comes to a stop. Because the acceleration is constant, the kinematic equations for the change in velocity is

$$v_{x,f}(t_f) - v_{p,1} = a_x t_f.$$

We can solve this equation for $t = t_f$, where $v_{x,f}(t_f) = 0$

$$t_f = -v_{p,1} / a_x t.$$

Then the position of the plane when it first comes to rest is

$$x_p(t_f) - x_p(0) = v_{p,1}t_f + \frac{1}{2}a_x t_f^2 = -\frac{1}{2}\frac{v_{p,1}^2}{a_x}. \qquad (10.9.27)$$

Then using $x_p(0) = 0$ and substituting Eq. (10.9.26) into Eq. (10.9.27) yields

$$x_p(t_f) = \frac{1}{2}\frac{(m_p + m_s)v_{p,1}^2}{(F_{g,p} + \mu_k m_s g)} . \qquad (10.9.28)$$

We now use the condition from conservation of the momentum law during the collision, Eq. (10.9.19) in Eq. (10.9.28) yielding

$$x_p(t_f) = \frac{m_p^2 v_{p,i}^2}{2(m_p + m_s)(F_{g,p} + \mu_k m_s g)} . \qquad (10.9.29)$$

Substituting the given values into Eq. (10.9.28) yields

$$x_p(t_f) = \frac{(1000\text{ kg})^2(40\text{ m}\cdot\text{s}^{-1})^2}{2(1000\text{ kg} + 120\text{ kg})(1400\text{ N} + (0.4)(120\text{ kg})(9.8\text{m}\cdot\text{s}^{-2}))} = 3.8\times 10^2\text{ m} . \qquad (10.9.30)$$