Chapter 26 Elastic Properties of Materials

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26.1 Introduction

In our study of rotational and translational motion of a rigid body, we assumed that the rigid body did not undergo any deformations due to the applied forces. Real objects deform when forces are applied. They can stretch, compress, twist, or break. For example when a force is applied to the ends of a wire and the wire stretches, the length of the wire increases. More generally, when a force per unit area, referred to as *stress,* is applied to an object, the particles in the object may undergo a relative displacement compared to their unstressed arrangement. *Strain* is a normalized measure of this deformation. For example, the *tensile strain* in the stretched wire is fractional change in length of a stressed wire. The stress may not only induce a change in length, but it may result in a volume change as occurs when an object is immersed in a fluid, and the fluid exerts a force per unit area that is perpendicular to the surface of the object resulting in a *volume strain* which is the fractional change in the volume of the object. Another type of stress, known as a *shear stress* occurs when forces are applied tangential to the surface of the object, resulting in a deformation of the object. For example, when scissors cut a thin material, the blades of the scissors exert shearing stresses on the material causing one side of the material to move down and the other side of the material to move up as shown in Figure 26.1, resulting in a *shear strain*. The material deforms until it ultimately breaks.

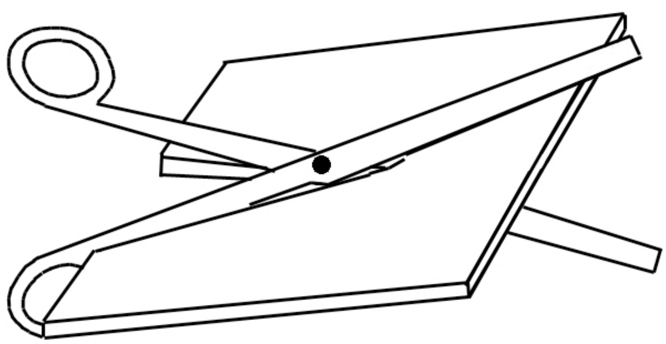

Figure 26.1: Scissors cutting a thin material[1]

In many materials, when the stress is small, the stress and strains are linearly proportional to one another. The material is then said to obey Hooke's Law. The ratio of stress to strain is called the *elastic modulus.* Hooke's Law only holds for a range of stresses, a range referred to as the *elastic region.* An *elastic body* is one in which Hooke's Law applies and when the applied stress is removed, the body returns to its initial shape. Our

[1] Mohsen Mahvash, et al, IEEE Trans Biomed Eng. 2008, March; 55(3); 848-856.

idealized spring is an example of an elastic body. Outside of the elastic region, the stress-strain relationship is non-linear until the object breaks.

26.2 Stress and Strain in Tension and Compression

Consider a rod with cross sectional area A and length l_0. Two forces of the same magnitude $F_\perp$ are applied perpendicularly at the two ends of the section stretching the rod to a length l (Figure 26.2), where the beam has been stretched by a positive amount $\delta l = l - l_0$.

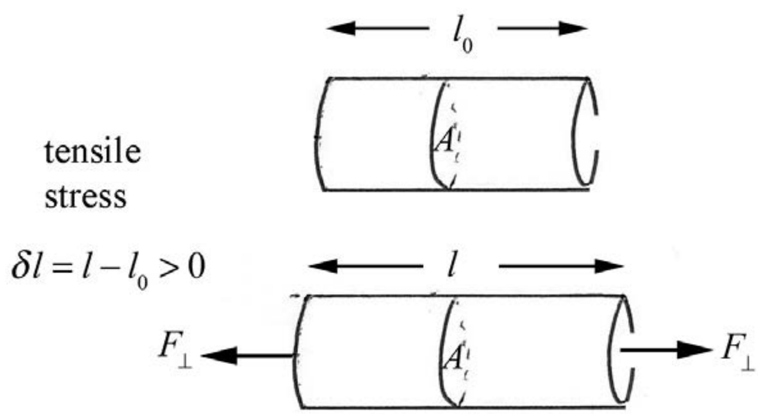

Figure 26.2: Tensile stress on a rod

The ratio of the applied perpendicular force to the cross-sectional area is called the *tensile stress*,

$$\sigma_T = \frac{F_\perp}{A}. \tag{26.2.1}$$

The ratio of the amount the section has stretched to the original length is called the *tensile strain*,

$$\varepsilon_T = \frac{\delta l}{l_0}. \tag{26.2.2}$$

Experimentally, for sufficiently small stresses, for many materials the stress and strain are linearly proportional,

$$\frac{F_\perp}{A} = Y\frac{\delta l}{l_0} \quad \text{(Hooke's Law)}. \tag{26.2.3}$$

where the constant of proportionality Y is called **Young's modulus**. The SI unit for Young's Modulus is the **pascal** where $1\,\text{Pa} \equiv 1\,\text{N} \cdot \text{m}^{-2}$. Note the following conversion factors between SI and English units: $1\ \text{bar} \equiv 10^5\ \text{Pa}$, $1\ \text{psi} \equiv 6.9 \times 10^{-2}\ \text{bar}$, and $1\ \text{bar} = 14.5\ \text{psi}$. In Table 26.1, Young's Modulus is tabulated for various materials. Figure 26.3 shows a plot of the stress-strain relationship for various human bones. For stresses greater than approximately $70\ \text{N} \cdot \text{mm}^{-2}$, the material is no longer elastic. At a certain point for each bone, the stress-strain relationship stops, representing the fracture point.

Material	Young's Modulus, *Y* (Pa)
Iron	21×10^{10}
Nickel	21×10^{10}
Steel	20×10^{10}
Copper	11×10^{10}
Brass	9.0×10^{10}
Aluminum	7.0×10^{10}
Crown Glass	6.0×10^{10}
Cortical Bone	$7 \times 10^9 - 30 \times 10^9$
Lead	1.6×10^{10}
Tendon	2×10^7
Rubber	$7 \times 10^5 - 40 \times 10^5$
Blood vessels	2×10^5

Table 26.1: Young's Modulus for various materials

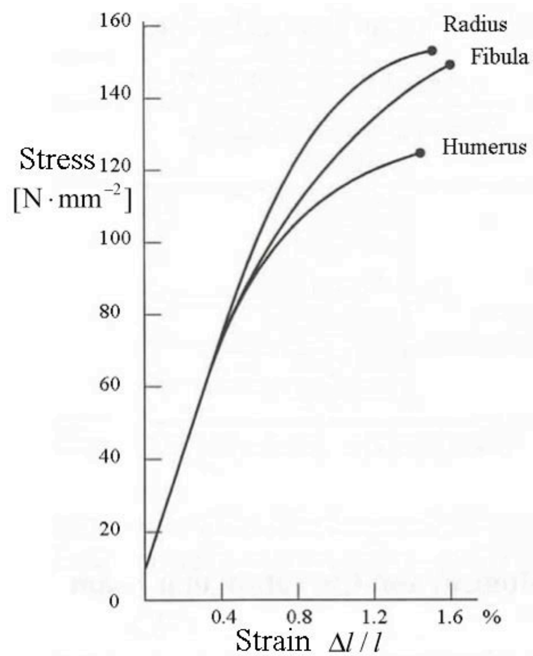

Figure 26.3: Stress-strain relation for various human bones (figure from H. Yamada, Strength of Biological Materials)

When the material is under compression, the forces on the ends are directed towards each other producing a *compressive stress* resulting in a *compressive strain* (Figure 26.4). For compressive strains, if we define $\delta l = l_0 - l > 0$ then Eq. (26.2.3) holds for compressive stresses provided the compressive stress is not too large. For many materials, Young's Modulus is the same when the material is under tension and compression. There are some important exceptions. Concrete and stone can undergo compressive stresses but fail when the same tensile stress is applied. When building with these materials, it is important to design the structure so that the stone or concrete is never under tensile stresses. Arches are used as an architectural structural element primarily for this reason.

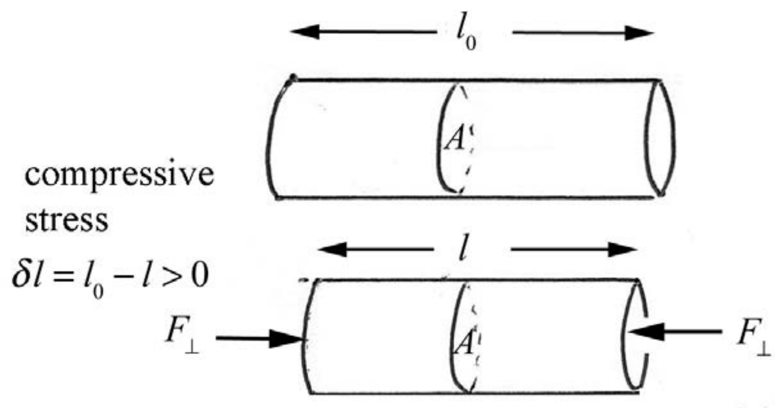

Figure 26.4: Compressive Stress

26.3 Shear Stress and Strain

The surface of material may also be subjected to tangential forces producing a shearing action. Consider a block of height h and area A, in which a tangential force, $\vec{\mathbf{F}}_{\tan}$, is applied to the upper surface. The lower surface is held fixed. The upper surface will shear by an angle α corresponding to a horizontal displacement δx. The geometry of the shearing action is shown in Figure 26.5.

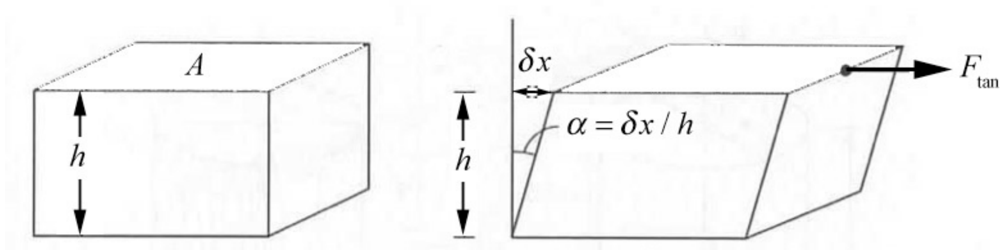

Figure 26.5: Shearing forces

The *shear stress* is defined to be the ratio of the tangential force to the cross sectional area of the surface upon which it acts,

$$\sigma_S = \frac{F_{\tan}}{A}. \tag{26.3.1}$$

The *shear strain* is defined to be the ratio of the horizontal displacement to the height of the block,

$$\alpha = \frac{\delta x}{h}. \tag{26.3.2}$$

For many materials, when the shear stress is sufficiently small, experiment shows that a Hooke's Law relationship holds in that the shear stress is proportional to shear strain,

$$\frac{F_{\tan}}{A} = S\frac{\delta x}{h} \quad \text{(Hooke's Law)}. \tag{26.3.3}$$

where the constant of proportional, S, is called the **shear modulus**. When the deformation angle is small, $\delta x / h = \tan\alpha \simeq \sin\alpha \simeq \alpha$, and Eq. (26.3.3) becomes

$$\frac{F_{\tan}}{A} \simeq S\alpha \quad \text{(Hooke's Law)}. \tag{26.3.4}$$

In Table 26.2, the shear modulus is tabulated for various materials.

Table 26.2: Shear Modulus for Various Materials

Material	Shear Modulus, S (Pa)
Nickel	7.8×10^{10}
Iron	7.7×10^{10}
Steel	7.5×10^{10}
Copper	4.4×10^{10}
Brass	3.5×10^{10}
Aluminum	2.5×10^{10}
Crown Glass	2.5×10^{10}
Lead	0.6×10^{10}
Rubber	$2\times10^{5}-10\times10^{5}$

Example 26.1: Stretched wire

An object of mass 1.5×10^{1} kg is hanging from one end of a steel wire. The wire without the mass has an unstretched length of 0.50 m. What is the resulting strain and elongation of the wire? The cross-sectional area of the wire is 1.4×10^{-2} cm^2.

Solution: When the hanging object is attached to the wire, the force at the end of the wire acting on the object exactly balances the gravitational force. Therefore by Newton's Third Law, the tensile force stressing the wire is

$$F_{\perp} = mg. \tag{26.3.5}$$

We can calculate the strain on the wire from Hooke's Law (Eq. (26.2.3)) and the value of Young's modulus for steel 20×10^{10} Pa (Table 26.1);

$$\frac{\delta l}{l_0} = \frac{F_\perp}{YA} = \frac{mg}{YA} = \frac{(1.5\times 10^1 \text{ kg})(9.8 \text{ m}\cdot\text{s}^{-2})}{(2.0\times 10^{11} \text{ Pa})(1.4\times 10^{-6} \text{ m}^2)} = 5.3\times 10^{-4}. \qquad (26.3.6)$$

The elongation δl of the wire is then

$$\delta l = \frac{mg}{YA} l_0 = (5.3\times 10^{-4})(0.50 \text{ m}) = 2.6\times 10^{-4} \text{ m}. \qquad (26.3.7)$$

26.4 Elastic and Plastic Deformation

Consider a single sheet of paper. If we bend the paper gently, and then release the constraining forces, the sheet will return to its initial state. This process of gently bending is reversible as the paper displays *elastic behavior*. The internal forces responsible for the deformation are conservative. Although we do not have a simple mathematical model for the potential energy, we know that mechanical energy is constant during the bending. We can take the same sheet of paper and crumple it. When we release the paper it will no longer return to its original sheet but will have a permanent deformation. The internal forces now include non-conservative forces and the mechanical energy is decreased. This *plastic behavior* is irreversible.

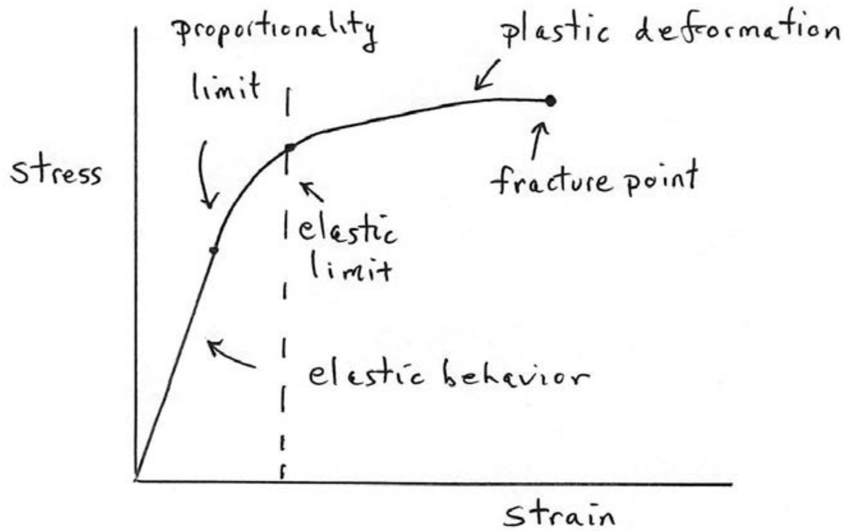

Figure 26.5: Stress-strain relationship

When the stress on a material is linearly proportional to the strain, the material behaves according to Hooke's Law. The *proportionality limit* is the maximum value of stress at which the material still satisfies Hooke's Law. If the stress is increased above the proportionality limit, the stress is no longer linearly proportional to the strain. However, if the stress is slowly removed then the material will still return to its original state; the material behaves elastically. If the stress is above the proportionality limit, but less then the elastic limit, then the stress is no longer linearly proportional to the strain. Even in this non-linear region, if the stress is slowly removed then the material will return to its original state. The maximum value of stress in which the material will still remain elastic is called the *elastic limit*. For stresses above the elastic limit, when the stress is removed the material will not return to its original state and some permanent deformation sets in, a state referred to as a *permanent set*. This behavior is referred to as *plastic deformation*. For a sufficiently large stress, the material will *fracture*. Figure 26.5 illustrates a typical stress-strain relationship for a material. The value of the stress that fractures a material is referred to as the *ultimate tensile strength.* The ultimate tensile strengths for various materials are listed in Table 26.3. The tensile strengths for wet human bones are for a person whose age is between 20 and 40 years old.

Table 26.3: Ultimate Tensile Strength for Various Materials

Material	Shear Modulus, *S* (Pa)
Femur	1.21×10^{8}
Humerus	1.22×10^{8}
Tibia	1.40×10^{8}
Fibula	1.46×10^{8}
Ulna	1.48×10^{8}
Radius	1.49×10^{8}
Aluminum	2.2×10^{8}
Iron	3.0×10^{8}
Brass	4.7×10^{8}
Steel	$5-20\times 10^{8}$

Example 26.2: Ultimate Tensile Strength of Bones

The ultimate tensile strength of the wet human tibia (for a person of age between 20 and 40 years) is $1.40\times 10^{8}\ \text{Pa}$. If a greater compressive force per area is applied to the tibia then the bone will break. The smallest cross sectional area of the tibia, about $3.2\ \text{cm}^{2}$, is slightly above the ankle. Suppose a person of mass $60\ \text{kg}$ jumps to the ground from a height $2.0\ \text{m}$ and absorbs the shock of hitting the ground by bending the knees. Assume that there is constant deceleration during the collision. During the collision, the person lowers her center of mass by an amount $d = 1.0\ \text{cm}$. (a) What is the collision time Δt_{col}? (b) Find the average force of the ground on the person during the collision. (c) Can we effectively ignore the gravitational force during the collision? (d) Will the person break her ankle? (e) What is the minimum distance $\Delta d_{\min}$ that she would need to lower her center of mass so she does not break her ankle? What is the ratio $h_0 / \Delta d_{\min}$? What factors does this ratio depend on?

Solution: Choose a coordinate system with the positive y-direction pointing up, and the origin at the ground. While the person is falling to the ground, mechanical energy is constant (we are neglecting any non-conservative work due to air resistance). Choose the contact point with the ground as the zero potential energy reference point. Then the initial mechanical energy is

$$E_0 = U_0 = mgh_0 . \tag{26.4.1}$$

The mechanical energy of the person just before contact with the ground is

$$E_b = K_1 = \frac{1}{2} m v_b{}^2 . \tag{26.4.2}$$

The constancy of mechanical energy implies that

$$mgh_0 = \frac{1}{2} m v_b{}^2 . \tag{26.4.3}$$

The speed of the person the instant contact is made with the ground is then

$$v_b = \sqrt{2gh_0} . \tag{26.4.4}$$

If we treat the person as the system then there are two external forces acting on the person, the gravitational force $\vec{\mathbf{F}}^g = -mg\hat{\mathbf{j}}$ and a normal force between the ground and the person $\vec{\mathbf{F}}^N = N\hat{\mathbf{j}}$. This force varies with time but we shall consider the time average $\vec{\mathbf{F}}^N_{\text{ave}} = N_{\text{ave}}\hat{\mathbf{j}}$. Then using Newton's Second Law,

$$N_{\text{ave}} - mg = ma_{y,\text{ave}} . \tag{26.4.5}$$

The y-component of the average acceleration is equal to

$$a_{y,\text{ave}} = \frac{N_{\text{ave}}}{m} - g . \tag{26.4.6}$$

Set $t = 0$ for the instant the person reaches the ground; then $v_{y,0} = -v_b$. The displacement of the person while in contact with the ground for the time interval Δt_{col} is given by

$$\Delta y = -v_b \Delta t_{\text{col}} + \frac{1}{2} a_{y,\text{ave}} \Delta t_{\text{col}}^2 . \tag{26.4.7}$$

The y-component of the velocity is zero at $t = \Delta t_{\text{col}}$ when the person's displacement is $\Delta y = -d$,

$$0 = -v_b + a_{y,\text{ave}} \Delta t_{\text{col}} . \tag{26.4.8}$$

Solving Eq. (26.4.8) for the collision time yields

$$\Delta t_{\text{col}} = v_b / a_{y,\text{ave}} . \tag{26.4.9}$$

We can now substitute $\Delta y = -d$, Eq. (26.4.9), and Eq. (26.4.4) into Eq. (26.4.7) and solve for the y-component of the acceleration, yielding

$$a_{y,\text{ave}} = \frac{gh_0}{d} . \tag{26.4.10}$$

We can solve for the collision time by substituting Eqs. (26.4.10) and Eq. (26.4.4) into Eq. (26.4.9) and using the given values in the problem statement, yielding

$$\Delta t_{\text{col}} = \frac{2d}{\sqrt{2gh_0}} = \frac{2(1.0 \times 10^{-2}\,\text{m})}{\sqrt{2(9.8\,\text{m} \cdot \text{s}^2)(2.0\,\text{m})}} = 3.2 \times 10^{-3}\,\text{s} . \tag{26.4.11}$$

Now substitute Eq. (26.4.10) for the y-component of the acceleration into Eq. (26.4.6) and solve for the average normal force

$$N_{\text{ave}} = mg\left(1 + \frac{h_0}{d}\right) = (60\,\text{kg})(9.8\,\text{m} \cdot \text{s}^{-2})\left(1 + \frac{(2.0\,\text{m})}{(1.0 \times 10^{-2}\,\text{m})}\right) = 1.2 \times 10^5\,\text{N} . \tag{26.4.12}$$

Notice that the factor $1 + h_0 / d \simeq h_0 / d$ so *during the collision* we can effectively ignore the external gravitational force. The average compressional force per area on the person's ankle is the average normal force divided by the cross sectional area

$$P = \frac{N_{\text{ave}}}{A} \simeq \frac{mg}{A}\left(\frac{h_0}{d}\right) = \frac{1.2 \times 10^5\,\text{N}}{3.2 \times 10^{-4}\,\text{m}^2} = 3.7 \times 10^8\,\text{Pa} . \tag{26.4.13}$$

From Table 26.3, the tensile strength of the tibia is $1.4 \times 10^8\,\text{Pa}$, so this fall is enough to break the tibia.

In order to find the minimum displacement that the center of mass must fall in order to avoid breaking the tibia bone, we set the force per area in Eq. (26.4.13) equal to $P = 1.4 \times 10^8\,\text{Pa}$. Because at this value of tensile strength,

$$\frac{PA}{mg} = \frac{(1.4 \times 10^{8}\ \text{Pa})((3.2 \times 10^{-4}\ \text{m}^{2})}{(60\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})} = 80 \qquad (26.4.14)$$

and so $PA >> mg$. We can solve Eq. (26.4.13) for the minimum displacement

$$d_{\min} = \frac{h_0}{\left(\dfrac{PA}{mg} - 1\right)} \simeq \frac{mgh_0}{PA} = \frac{(60\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})(2.0\ \text{m})}{(1.4 \times 10^{8}\ \text{Pa})(3.2 \times 10^{-4}\ \text{m}^{2})} = 2.6\ \text{cm}, \qquad (26.4.15)$$

where we used the fact that

$$\frac{PA}{mg} = \frac{(1.4 \times 10^{8}\ \text{Pa})((3.2 \times 10^{-4}\ \text{m}^{2})}{(60\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})} = 76 \qquad (26.4.16)$$

and so $PA >> mg$. The ratio

$$h_0 / d_{\min} \simeq PA / mg = 76\,. \qquad (26.4.17)$$

This ratio depends on the compressive strength of the bone, the cross sectional area, and inversely on the weight of the person. The maximum normal force is anywhere from two to ten times the average normal force. A safe distance to lower the center of mass would be about 20 cm.

26.5 Bending Beam

Consider a section of a beam that is under both compression and tension and bends as shown in Figure 26.6.

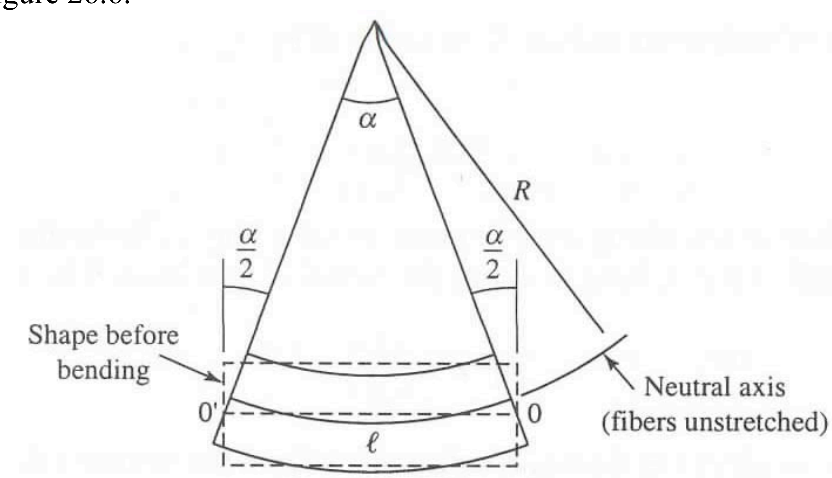

Figure 26.6: Beam under both compression and tension

The upper portion of the beam is under compression and the lower portion of the beam is under tension. The centerline is neither under tension or compression and is called the *neutral axis*. For the section shown in Figure 26.6, the neutral axis has length l.

We shall compute the strain at a distance x from neutral axis. The geometry of the deformed beam is shown in Figure 26.7.

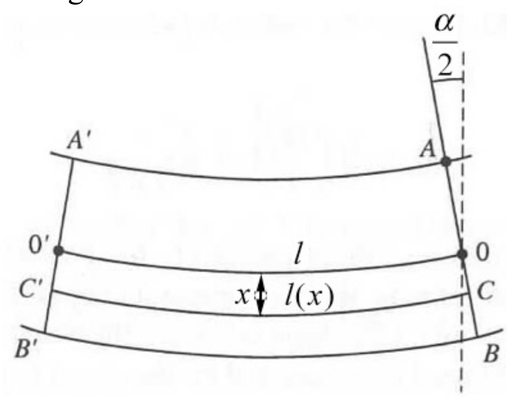

Figure 26.7: Deformed beam with neutral line

At a distance x from the neutral line, the beam stretches on each side by an amount $y = x\tan(\alpha/2)$. For small angles of deformation, $\tan(\alpha/2) \simeq (\alpha/2)$. Therefore the length of the section of the beam at a distance x from the neutral axis is

$$l(x) = l + 2x\tan(\alpha/2) \simeq l + 2x(\alpha/2) = l + x\alpha. \qquad (26.5.1)$$

The amount the beam has stretched is given by $\delta l = x\alpha$. The arc length is related to the *radius of curvature*, R, by $l = R\alpha$, therefore the stretched length is $\delta l = xl/R$. The strain as a function of distance x from the neutral line is given by

$$\frac{\delta l}{l} = \frac{x}{R}. \qquad (26.5.2).$$

At each point along a line lying a distance x from the neutral line, the stress is therefore

$$\sigma = \frac{F_{\perp}}{A} = Y\frac{\delta l}{l} = Y\frac{x}{R}. \qquad (26.5.3)$$

The forces these internal stresses exert on the line AOB are shown in Figure 26.8. These internal stresses produce an internal torque about a point O lying on the neutral line. This internal torque is called the *bending moment*.

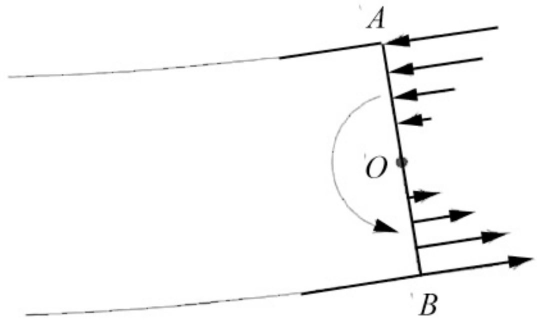

Figure 26.8: Internal torque about a point lying on the neutral line

Consider a strip of cross-sectional strip of area dA, a distance x from the neutral line (Figure 26.9).

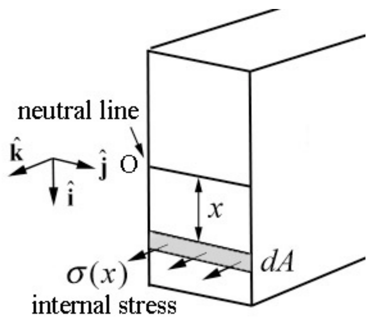

Figure 26.9: Internal stress due to deformation of body

The total force acting on the strip is

$$dF = \sigma\, dA = \frac{Y}{R} x dA\,. \tag{26.5.4}$$

The internal torque about the neutral line is

$$d\vec{\boldsymbol{\tau}} = x\,\hat{\mathbf{i}} \times dF\,\hat{\mathbf{k}} = -\frac{Y}{R} x^2 dA \hat{\mathbf{j}}\,. \tag{26.5.5}$$

The *bending moment* is then the integral of the internal torque over the cross-section and is given by

$$\vec{\mathbf{M}}_B = \int_{area} d\vec{\boldsymbol{\tau}} = -\frac{Y}{R} \int_{area} x^2\, dA\,\hat{\mathbf{j}} = -\frac{Y}{R} I_A \hat{\mathbf{j}}\,, \tag{26.5.6}$$

where

$$I_A = \int_{area} x^2\, dA\,. \tag{26.5.7}$$

is called the *area moment of inertia* of the beam. When the radius of curvature, R, is large, the magnitude of the bending moment, M_B, is small; when the radius of curvature is small, the bending moment is large. The *curvature* of the beam is defined as

$$K \equiv \frac{1}{R} = \frac{M_B}{YI_A}. \tag{26.5.8}$$

Example 26.3: Bending Moment of a Beam

Calculating the area moment of inertia of a 2 by 6 inch beam, which has a width of $w = 38\text{ mm}$ and height of $h = 140\text{ mm}$, about the stiff vertical direction shown in Figure 26.10.

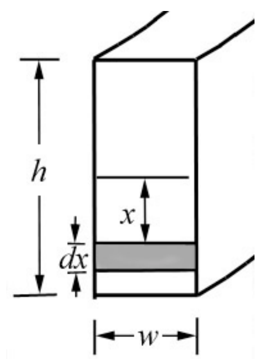

Figure 26.10: cross section of a 2 by 6 inch beam.

Solution: The cross-sectional area element $dA = wdx$. The area moment of inertia is

$$\begin{aligned} I_A &= w \int_{x'=-h/2}^{x'=h/2} x'^2\, dx' = w\frac{x'^3}{3}\bigg|_{x'=-h/2}^{x'=h/2} = w\frac{(h/2)^3}{3} - w\frac{(-h/2)^3}{3} = w\frac{h^3}{12} \\ &= (38\text{ mm})\frac{(140\text{ mm})^3}{12} = 8.7\times10^{-6}\text{ m}^3. \end{aligned} \tag{26.5.9}$$

The bending moment is proportional to the cube of the height h of the beam in the plane of the bending (Figure 26.11) and only linearly proportional to the width w perpendicular to that plane.

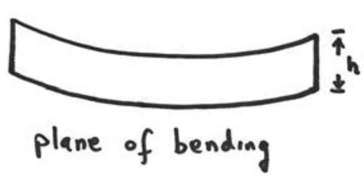

Figure 26.11: Plane of bending about the stiff direction of a 2 by 6 beam

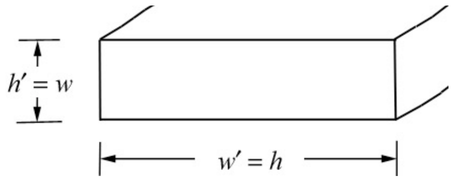

Figure 26.12: Bending about the more flexible dimension of a 2 by 6 beam

The bending moment about the plane in which the dimensions are interchanged, $h' = w$ and $w' = h$ is smaller (Figure 26.12),

$$I'_A = \frac{w'h'^3}{12} = \frac{hw^3}{12} = (140\text{ mm})\frac{(38\text{ mm})^3}{12} = 6.4\times10^{-7}\text{ m}^3. \qquad (26.5.10)$$

Note that

$$I'_A / I_A = \left(\frac{w}{h}\right)^2 = \left(\frac{38}{140}\right)^2 = 7.4\times10^{-2} \qquad (26.5.11)$$

Because the curvature is proportional to the bending moment, the greater the bending moment, the greater the deformation. The cross sections of various structural elements are shown in Figure 26.13. The I-beam provides the desired stiffness and minimizes the amount of material as well.

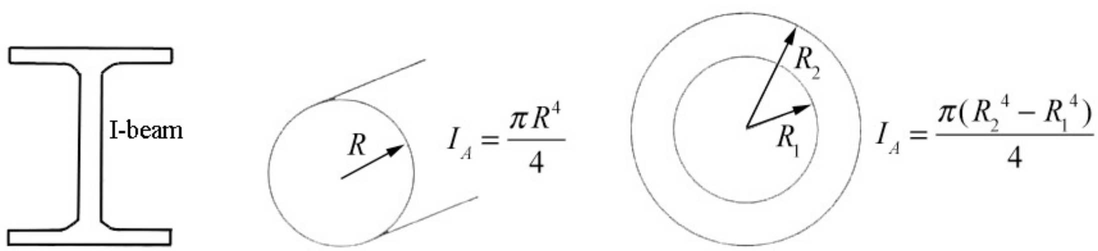

Figure 26.13: Various structural elements

26.6 Differential Equation for Deflection of Loaded Beam

We shall now solve for a differential equation that describes how the neutral axis changes as a function of distance along the axis when a stress is applied to the beam

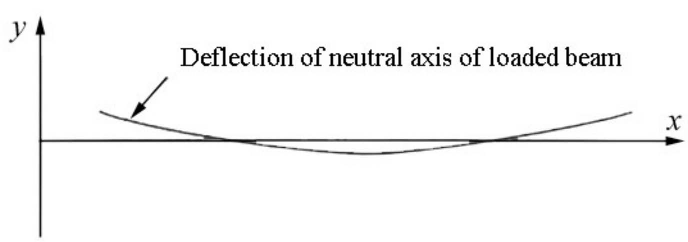

Figure 26.14: Deflection of neutral axis of loaded beam

Consider a beam that deflects under the action of a load and the neutral line describes some curve $y(x)$ (Figure 26.14). We first begin by finding an expression for the curvature $K \equiv 1/R$ in terms of the second derivative of the equation for the curve describing the shape of the beam.

Let $\theta(x)$ describe the angle that the beam is bent with respect to the x-axis at the point x (Figure 26.15). We shall consider cases in which the angle $\theta(x)$ is small and make the approximation that $\tan\theta(x) \simeq \theta(x)$.

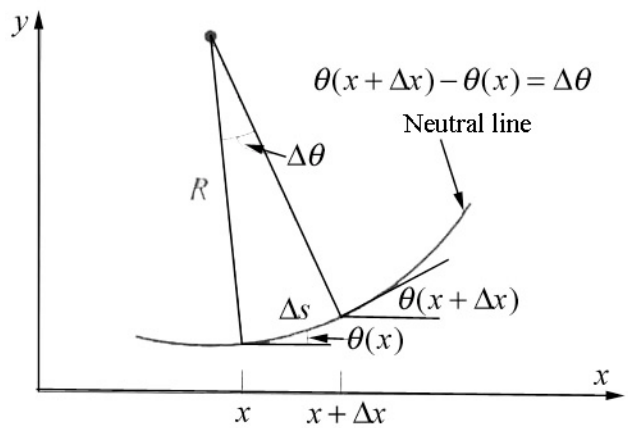

Figure 26.15: Differential analysis of bending

The arc length is given by $\Delta s = R\Delta\theta$. The slope of the curve $y(x)$ at the two points x and $x+\Delta x$ are given by

$$\frac{dy}{dx}(x) = \tan\theta(x) \simeq \theta(x), \tag{26.5.12}$$

and

$$\frac{dy}{dx}(x+\Delta x) = \tan\theta(x+\Delta x) \simeq \theta(x+\Delta x) = \theta(x)+\Delta\theta. \tag{26.5.13}$$

Therefore the second derivative is given by

$$\frac{d^2y}{dx^2}(x) = \lim_{\Delta x\to 0}\frac{\frac{dy}{dx}(x+\Delta x)-\frac{dy}{dx}(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{(\theta(x)+\Delta\theta)-\theta(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta\theta}{\Delta x}. \tag{26.5.14}$$

The arc length is

$$\Delta s = R\Delta\theta = (\Delta x^2+\Delta y^2)^{1/2} = \Delta x\left(1+\left(\frac{\Delta y}{\Delta x}\right)^2\right)^{1/2}. \tag{26.5.15}$$

Thus

$$\frac{\Delta\theta}{\Delta x} = \frac{1}{R}\left(1+\left(\frac{\Delta y}{\Delta x}\right)^2\right)^{1/2} . \qquad (26.5.16).$$

Therefore the second derivative is given by

$$\frac{d^2 y}{dx^2}(x) = \lim_{\Delta x \to 0}\frac{\Delta\theta}{\Delta x} = \lim_{\Delta x \to 0}\frac{1}{R}\left(1+\left(\frac{\Delta y}{\Delta x}\right)^2\right)^{1/2} = \frac{1}{R}\left(1+\left(\frac{dy}{dx}(x)\right)^2\right)^{1/2} . \qquad (26.5.17)$$

For small deflections,

$$\frac{dy}{dx}(x) \simeq 0 . \qquad (26.5.18)$$

In that case, the second derivative is equal to the curvature

$$\frac{d^2 y}{dx^2} \simeq \frac{1}{R} = K . \qquad (26.5.19)$$

We can now determine the differential equation for the loaded beam by applying our result for the curvature (Eq. (26.5.8)),

$$\frac{d^2 y}{dx^2} = \frac{M_B(x)}{YI_A} . \qquad (26.5.20)$$

26.7 Cantilevered Beam

Consider a beam that at $x = 0$ is held fixed and allowed to bend under a load F applied at the other end, $x = L$ (Figure 26.16).

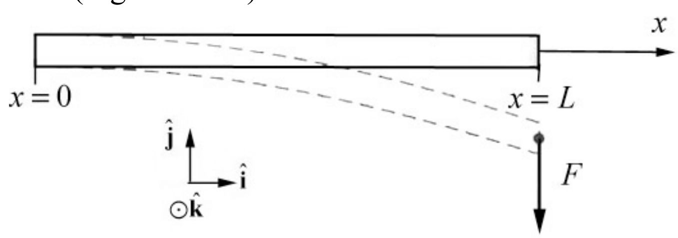

Figure 26.16: Loaded beam with one end held fixed

Consider a mathematical cut in the beam a distance x from the fixed end. The right side of the beam is static so there must be a vertical force $\vec{\mathbf{F}}(x) = F\hat{\mathbf{j}}$ at x to balance the load $\vec{\mathbf{F}}_L = -F\hat{\mathbf{j}}$ at $x = L$ (Figure 26.17).

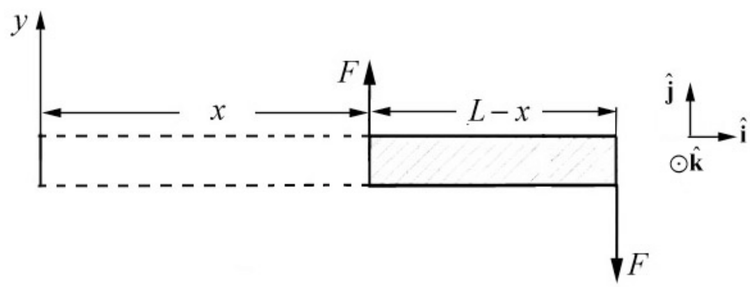

Figure 26.17: Internal vertical force at imaginary cut at a distance x from the fixed end of beam

In addition to the shearing vertical force, there are compressive and tensile forces at x in order to balance the external torque about x (Figure 26.18), given by

$$\vec{\boldsymbol{\tau}}_L(x) = (L-x)\,\hat{\mathbf{i}} \times \vec{\mathbf{F}}_L = (L-x)\,\hat{\mathbf{i}} \times (-F\hat{\mathbf{j}}) = -(L-x)F\hat{\mathbf{k}}\,. \qquad (26.5.21)$$

The internal torque due to the compression and extension at x, which exactly balances the external torque is

$$\vec{\boldsymbol{\tau}}_{\text{int}}(x) = (L-x)F\hat{\mathbf{k}}\,. \qquad (26.5.22)$$

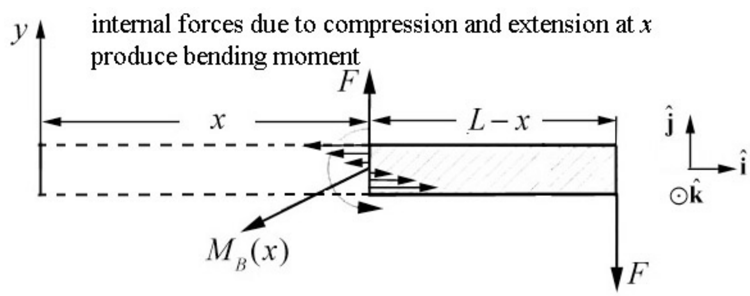

Figure 26.18: Bending moment for a loaded beam fixed at one end.

Because the beam is deflecting downwards, the bending moment in this example must be negative in order to produce negative curvature,

$$M_B = -(L-x)F\,. \qquad (26.5.23)$$

The bending moment as a function of distance from the fixed end is shown in Figure 26.19.

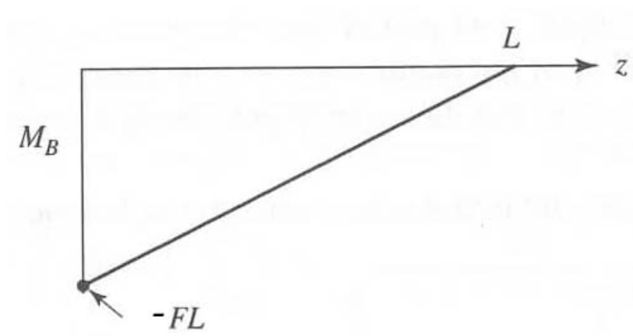

Figure 26.19: Bending moment

The differential equation (Eq. (26.5.20)) for the beam is

$$\frac{d^2y}{dx^2} = -\frac{F}{YI_A}(L-x). \tag{26.5.24}$$

The first integral of this equation is

$$\frac{dy}{dx} = \frac{F}{2YI_A}(L-x)^2 + a_1, \tag{26.5.25}$$

where a_1 is a constant of integration. The second integral is then

$$y(x) = -\frac{F}{6YI_A}(L-x)^3 + a_1x + a_2, \tag{26.5.26}$$

where a_2 is another constant of integration.

We can determine a_1 and a_2 by applying the boundary conditions that

1) At $x = 0$: $y = 0$, there is zero deflection at the fixed end, therefore

$$y(x=0) = -\frac{FL^3}{6YI_A} + a_2 \Rightarrow a_2 = \frac{FL^3}{6YI_A}, \tag{26.5.27}$$

2) At $x = 0$: $\frac{dy}{dx}(x=0) = 0$, the beam is level at the fixed end, therefore

$$\frac{dy}{dx}(0) = \frac{FL^2}{2YI_A} + a_1 \Rightarrow a_1 = -\frac{FL^2}{2YI_A}, \tag{26.5.28}$$

The general equation for the beam deflection is given by

$$y(x) = -\frac{F}{6YI_A}(L-x)^3 - \frac{FL^2}{2YI_A}x + \frac{FL^3}{6YI_A}, \qquad (26.5.29)$$

In particular at the loaded end, $x = L$, the deflection is

$$y(x = L) = -\frac{FL^3}{2YI_A} + \frac{FL^3}{6YI_A} = -\frac{FL^3}{3YI_A}, \qquad (26.5.30)$$

Example 26.4: Deflection of a Beam

Consider an eight foot long, 2 by 6 beam. The length of the beam is $L = 2.44\,\text{m}$. Recall The area moment of inertia about the stiff direction (Figure 26.10 in Example 26.3) is $I_A = 8.7\times10^{-6}\ \text{m}^3$. Young's modulus for wood is $Y = 1.0\times10^{10}\,\text{Pa}$. If we load the beam with a force of magnitude $F = 2.2\times10^2\ \text{N}$, then the beam will deflect at the loaded end by

$$y(x = L) = -\frac{FL^3}{3YI_A} = -\frac{(2.2\times10^2\ \text{N})(2.44\ \text{m})^3}{3(1.0\times10^{10}\,\text{Pa})(8.7\times10^{-6}\ \text{m}^3)} = 1.2\times10^{-2}\ \text{m} = 1.2\ \text{cm}\,. \qquad (26.5.31)$$

If we hang the load in the flexible direction (Figure 26.12), with, $I'_A = 6.4\times10^{-7}\ \text{m}^3$, then the beam will bend at the loaded by

$$y(x = L) = -\frac{FL^3}{3YI'_A} = -\frac{(2.2\times10^2\ \text{N})(2.44\ \text{m})^3}{3(1.0\times10^{10}\,\text{Pa})(6.4\times10^{-7}\ \text{m}^3)} = 1.7\times10^{-1}\ \text{m} = 17\ \text{cm}\,. \qquad (26.5.32)$$

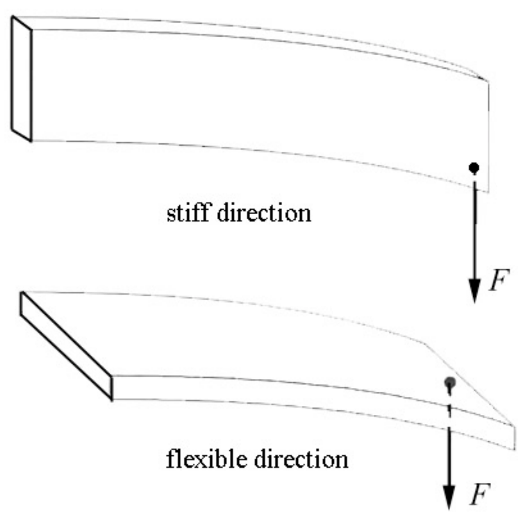

Figure 26.21: Loaded beam in stiff and flexible directions