Chapter 30 Navier-Stokes Equations

Chapter 29 Navier-Stokes Equations

29.1 Introduction

29.2 Acceleration Vector Field

Consider the path of a fluid particle, which we shall designate by the label 1, as shown in the figure below when the particle is located at the point with coordinates (x,y,z,t). The position vector of the fluid particle is given by

$$\vec{r}_1(x,y,z,t) = x_1(x,y,z,t)\hat{\mathbf{i}} + y_1(x,y,z,t)\hat{\mathbf{j}} + z_1(x,y,z,t)\hat{\mathbf{k}} . \qquad (29.1.1)$$

where $x_1(x,y,z,t)$, $y_1(x,y,z,t)$, and $z_1(x,y,z,t)$ are the component functions of the particle. If the fluid flow is described by the velocity field $\vec{v}(x,y,z,t)$, then the velocity of particle 1 at the point (x,y,z,t) is given by

$$\vec{v}_1(x,y,z,t) = v_{1x}(x,y,z,t)\hat{\mathbf{i}} + v_{1y}(x,y,z,t)\hat{\mathbf{j}} + v_{1z}(x,y,z,t)\hat{\mathbf{k}} . \qquad (29.1.2)$$

The acceleration of the particle can be found by differentiating the velocity. The velocity vector of the particle is a function of both time and the position of particle 1. Therefore the change of velocity of particle 1 can depend on how the velocity is changing in time and how the position of the particle is changing in space. In order to account for both of these changes, we need the chain rule for differentiation. The acceleration of particle 1 is then

$$\vec{\mathbf{a}}_1 = \frac{d\vec{v}_1}{dt} = \frac{\partial \vec{v}_1}{\partial t} + \frac{\partial \vec{v}_1}{\partial x}\frac{dx_1}{dt} + \frac{\partial \vec{v}_1}{\partial y}\frac{dy_1}{dt} + \frac{\partial \vec{v}_1}{\partial z}\frac{dz_1}{dt} , \qquad (29.1.3)$$

where we have suppressed the reference to the coordinates (x,y,z,t). In the above expression for the acceleration, the derivatives of the coordinate position functions of particle 1 are just the respective component functions of the velocity of particle 1,

$$v_{1x} = \frac{dx_1}{dt}, \ v_{1y} = \frac{dy_1}{dt}, \ v_{1z} = \frac{dz_1}{dt} . \qquad (29.1.4)$$

Therefore the acceleration of particle 1 is

$$\vec{\mathbf{a}}_1 = \frac{d\vec{v}_1}{dt} = \frac{\partial \vec{v}_1}{\partial t} + \frac{\partial \vec{v}_1}{\partial x}v_{1x} + \frac{\partial \vec{v}_1}{\partial y}v_{1y} + \frac{\partial \vec{v}_1}{\partial z}v_{1z} . \qquad (29.1.5)$$

Eq. (29.1.5) applies to all fluid particles, so we can drop the reference to particle 1, and therefore the acceleration vector field associated with the flow is given by

$$\vec{\mathbf{a}}(x,y,z,t) = \frac{d\vec{\mathbf{v}}}{dt} = \frac{\partial \vec{\mathbf{v}}}{\partial t} + \frac{\partial \vec{\mathbf{v}}}{\partial x} v_x + \frac{\partial \vec{\mathbf{v}}}{\partial y} v_y + \frac{\partial \vec{\mathbf{v}}}{\partial z} v_z . \tag{29.1.6}$$

Note that the $d\vec{\mathbf{v}}/dt$ does not correspond to the rate of change of the velocity field at a fixed point in space, but corresponds to the rate of change of the velocity of a fluid particle as it moves about it space. The first term corresponds to how the velocity is changed at a fixed point in space during a small time interval dt. The second term corresponds to the difference in velocities at the same instant in time between two points in space that are connected by the displacement $d\vec{\mathbf{r}}$ of the fluid particle in the time interval dt.

Recall that the gradient partial differential vector operator is defined in Cartesian coordinates by the expression

$$\vec{\nabla} = \frac{\partial}{\partial x} \hat{\mathbf{i}} + \frac{\partial}{\partial y} \hat{\mathbf{j}} + \frac{\partial}{\partial z} \hat{\mathbf{k}} . \tag{29.1.7}$$

In particular, consider the scalar partial differential operator given by

$$\begin{aligned} \vec{\mathbf{v}} \cdot \vec{\nabla} &= \left(v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}} \right) \cdot \left(\frac{\partial}{\partial x} \hat{\mathbf{i}} + \frac{\partial}{\partial y} \hat{\mathbf{j}} + \frac{\partial}{\partial z} \hat{\mathbf{k}} \right) \\ &= v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \end{aligned} \tag{29.1.8}$$

It acts on functions. For example, the action of this operator on the components of the velocity vector field are

$$\begin{aligned} (\vec{\mathbf{v}} \cdot \vec{\nabla}) v_x &= v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}, \\ (\vec{\mathbf{v}} \cdot \vec{\nabla}) v_y &= v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}, \\ (\vec{\mathbf{v}} \cdot \vec{\nabla}) v_z &= v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}. \end{aligned} \tag{29.1.9}$$

We can combine these three scalar expressions into one vector expression

$$(\vec{\mathbf{v}} \cdot \vec{\nabla}) \vec{\mathbf{v}} = v_x \frac{\partial \vec{\mathbf{v}}}{\partial x} + v_y \frac{\partial \vec{\mathbf{v}}}{\partial y} + v_z \frac{\partial \vec{\mathbf{v}}}{\partial z} . \tag{29.1.10}$$

The expression in Eq.(29.1.10) measures the difference in velocity at a given instant in time between two points in space that are connected by the displacement $d\vec{r}$ of the fluid particle in the time interval dt.

The vector expression for the acceleration field (Eq. (29.1.6) can now be rewritten as

$$\vec{a}(x,y,z,t) = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \; . \tag{29.1.11}$$

The derivative operator

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\vec{v} \cdot \vec{\nabla}) = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \; . \tag{29.1.12}$$

is called the *material derivative*. The term $\vec{v} \cdot \vec{\nabla}$ is called the *convective derivative*. The expression for the acceleration field in terms of the material derivative is given by

$$\vec{a}(x,y,z,t) = \frac{D\vec{v}}{Dt} \equiv \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \; . \tag{29.1.13}$$

The material derivative can be applied to any variable that changes in time for a given particle (as seen by an observer moving along with the particle). For example for a given flow $\vec{v}(x,y,z,t)$ with varying pressure $P(x,y,z,t)$, the rate of change of pressure of a fluid particle is

$$\frac{DP}{Dt} = \frac{\partial}{\partial t} + (\vec{v} \cdot \vec{\nabla})P = \frac{\partial P}{\partial t} + v_x \frac{\partial P}{\partial x} + v_y \frac{\partial P}{\partial y} + v_z \frac{\partial P}{\partial z} \; . \tag{29.1.14}$$

29.4 Fluid Kinematics

Consider a small cubic volume element of fluid as shown in the figure below. The motion of the fluid between time t and time $t + \delta t$ may consist of a combination of different motions. The fluid element may translate through space while maintaining its shape. The element may undergo a linear deformation resulting in a volume change. The element may undergo a rotation. The element may undergo angular deformation resulting in a change in shape. The general motion is some complicated combination of all of these individual motions. We shall begin by studying each motion separately.

29.4.1 Translational Motion

Consider the point P located in the fluid element at time t undergoing a uniform flow $\vec{v}(x,y,z,t)$, which means that all spatial derivatives of the components of the velocity are zero. The fluid element will simply translate in the direction of the velocity and not undergo any change in shape.

29.4.2 Linear Deformation

Now assume that the velocity field is non-uniform. For simplicity, let's assume that $\vec{\mathbf{v}}(x,y,z,t) = v_x(x,t)\hat{\mathbf{i}}$; there is only a non-zero x-component of the velocity and the only non-zero change is in the x-direction. Consider a fluid element of volume $\delta V = \delta x \delta y \delta z$, then the fluid element will undergo a change in volume. Consider the segments AB and CD of the fluid at time t. At time $t + \delta t$ the fluid element has both translated and deformed. As it moves, segment AB has moved to the position $x + v_x(x)\delta t$. Segment CD has moved to the position $x + \delta x + v_x(x + \delta x)\delta t$. The change in volume of the element is then

$$\Delta(\delta V) = \left(\left((x + \delta x + v_x(x + \delta x)\delta t) - (x + v_x(x)\delta t)\right)\delta y \delta z\right) - \delta x \delta y \delta z$$
$$= ((v_x(x + \delta x) - v_x(x))\delta t)\delta y \delta z. \tag{29.2.1}$$

Now apply the Taylor formula to the x-component of the velocity keeping terms of first order in δx with the result that the change

$$v_x(x + \delta x) - v_x(x) = \frac{\partial v_x}{\partial x}\delta x. \tag{29.2.2}$$

Then the change in the volume element is

$$\Delta(\delta V) = \frac{\partial v_x}{\partial x}\delta t\,\delta x \delta y \delta z = \frac{\partial v_x}{\partial x}\delta t \delta V. \tag{29.2.3}$$

The rate that the volume of the fluid is changing divided by the original volume of the fluid element is called the *volumetric dilatation rate* and is given by

$$\frac{1}{\delta V}\lim_{\delta t \to 0}\frac{\Delta(\delta V)}{\delta t} = \frac{\partial v_x}{\partial x}. \tag{29.2.4}$$

If there are non-zero changes in the component of the velocity in the two other directions then the volume dilatation rate can be generalized to the expression

$$\frac{1}{\delta V}\lim_{\delta t \to 0}\frac{\Delta(\delta V)}{\delta t} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \vec{\nabla}\cdot\vec{\mathbf{v}}. \tag{29.2.5}$$

Recall that the density of an incompressible fluid cannot change. Therefore there can be no volume change only translational displacement of the fluid element. Therefore the velocity field for an incompressible fluid must satisfy the condition that

$$\vec{\nabla} \cdot \vec{v} = 0, \quad \text{(incompressible fluid)} .$$

(29.2.6)

29.4.3 Rotational Motion and Angular Deformation

For simplicity we will begin by considering a two-dimensional flow with

$$\vec{v}(x,y,z,t) = v_x(y,t)\hat{\mathbf{i}} + v_y(x,t)\hat{\mathbf{j}} .$$

(29.2.7)

.

The non-zero derivative of the velocity components are

$$\frac{\partial v_x(y,t)}{\partial y} \neq 0 \quad \text{and} \quad \frac{\partial v_x(y,t)}{\partial y} \neq 0 .$$

(29.2.8)

The line segments AB and CD of the fluid will now undergo a rotation as shown in the figure below. The point A located at time t at (x,y) will move to the new point $(x + v_x(y)\delta t, y + v_y(x)\delta t)$. The change in position of point A is therefore

$$\Delta r_A = (x + v_x(y)\delta t, y + v_y(x)\delta t) - (x,y)$$
$$= (v_x(y)\delta t, v_y(x)\delta t).$$

(29.2.9)

The point B located at time t at $(x, y+\delta y)$ will move to the new point located at $(x + v_x(y+\delta y)\delta t, y + \delta y + v_y(x)\delta t)$. The change in position of point B is

$$\Delta r_B = (x + v_x(y+\delta y)\delta t, y + \delta y + v_y(x)\delta t) - (x, y+\delta y)$$
$$= (v_x(y+\delta y)\delta t, v_y(x)\delta t).$$

(29.2.10)

Notice that both points A and B translate in the y-direction by the same amount but point B has translating an additional distance in the x-direction given by

$$\Delta r_{B,x} - \Delta r_{A,x} = (v_x(y+\delta y) - v_x(y))\delta t = \frac{\partial v_x}{\partial y}\delta y \delta t .$$

(29.2.11)

Line segment AB has undergone a rotation with angular displacement

$$\Delta \theta_{AB} = \frac{\Delta r_{B,x} - \Delta r_{A,x}}{\delta y} = \frac{\partial v_x}{\partial y}\delta t .$$

(29.2.12)

.

The rate of angular displacement is clockwise and given by

$$\omega_{AB} = \lim_{\delta t \to 0} \frac{\Delta \theta_{AB}}{\delta t} = \frac{\partial v_x}{\partial y} \ .$$

(29.2.13)

We can make a similar calculation (try this as an exercise) for the line segment CD and shown that the rate of angular displacement is counterclockwise and given by

$$\omega_{CD} = \lim_{\delta t \to 0} \frac{\Delta \theta_{CD}}{\delta t} = \frac{\partial v_y}{\partial x} \ .$$

(29.2.14)

If we define the angular velocity, $\vec{\omega} = \omega_z \hat{\mathbf{k}}$, as the average of these two rotations, where clockwise rotations have a negative component and counterclockwise rotations have a positive component, then the z-component of the angular velocity is given by

$$\omega_z = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right).$$

(29.2.15)

We can generalize this argument to three dimensional flow where

$$\omega_x = \frac{1}{2}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right),$$
$$\omega_y = \frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right).$$

(29.2.16)

In general, the angular rotation can be written as a vector product

$$\vec{\omega} = \frac{1}{2}(\vec{\nabla} \times \vec{\mathbf{v}})$$

(29.2.17)

The **vorticity** of the velocity flow is defined to be the twice the angular rotation vector

$$\vec{\zeta} = 2\vec{\omega} = (\vec{\nabla} \times \vec{\mathbf{v}})$$

(29.2.18)

For our two-dimensional flow, $\omega_z = 0$ when $\partial v_y / \partial x = -\partial v_x / \partial y$; the fluid will only undergo pure rotation. When $\partial v_y / \partial x = \partial v_x / \partial y$, then $\omega_z = 0$, and the fluid will only undergo angular deformation and not rotate. For the general case, if the velocity field satisfies the condition that

$$\vec{\nabla} \times \vec{\mathbf{v}} = \vec{\mathbf{0}} \quad \text{(irrotational)},$$

(29.2.19)

then the flow is said to be *irrotational*.

29.5 Forces on Fluid Element

29.5.1 Surfaces Forces

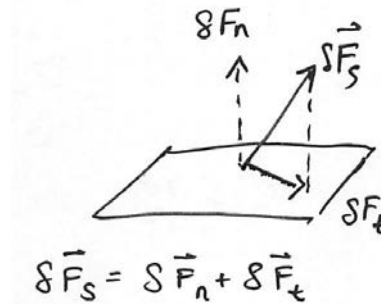Consider the forces acting on the surface of a fluid element as shown in Figure 29. .

Figure 29. Forces on the surface of a fluid volume element

The surface force can be decomposed into normal and tangential components,

$$\delta \vec{F}_s = \delta \vec{F}_n + \delta \vec{F}_t .$$
(29.3.1)

Let F_n denote the magnitude of the normal force. Let δA denote the area of the surface element. The magnitude of the *normal stress* is defined to be

$$\sigma_n = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A} .$$
(29.3.2)

We shall assign a sign for the normal stress as follows. For each surface on the fluid element, let $\hat{\mathbf{n}}_{out}$ denote the outward pointing unit normal on that surface. The normal stress is defined to be positive if $\delta \vec{F}_n \cdot \hat{\mathbf{n}}_{out} > 0$, and negative if $\delta \vec{F}_n \cdot \hat{\mathbf{n}}_{out} < 0$.

The magnitude of the shear stress is defined to be

$$\tau_t = \lim_{\delta A \to 0} \frac{\delta F_t}{\delta A} .$$
(29.3.3)

Choose a set of unit vectors $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$ on the surface such that $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{n}}_{out}$. Decompose the surface tangential force into components with respect to these surface unit vectors

$$\delta \vec{F}_t = \delta \vec{F}_{e_1} + \delta \vec{F}_{e_2} . \tag{29.3.4}$$

Then the shear stress on the surface has two components with magnitude

$$\tau_1 = \lim_{\delta A \to 0} \frac{\delta F_{e_1}}{\delta A} . \tag{29.3.5}$$

$$\tau_2 = \lim_{\delta A \to 0} \frac{\delta F_{e_2}}{\delta A} . \tag{29.3.6}$$

Our sign convention for the shear stress is that $\tau_1 > 0$ if $\delta \vec{F}_{e_1} \cdot \hat{e}_1 > 0$, and $\tau_2 > 0$ if $\delta \vec{F}_{e_2} \cdot \hat{e}_2 > 0$. Because there are three pairs of surfaces on the fluid element, there are nine different stresses. We shall introduce the following notation. Suppose we choose a cubic fluid element aligned with the Cartesian axes. Consider the plane defined by $x = x_0$. Let $\sigma_{xx}(x_0)$ denote the normal stress, $\sigma_{xy}(x_0)$ denote the shear stress in the y-direction, and $\sigma_{xz}(x_0)$ denote the shear stress in the z-direction (Figure 29.).
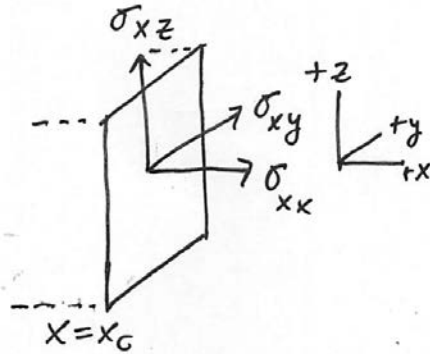
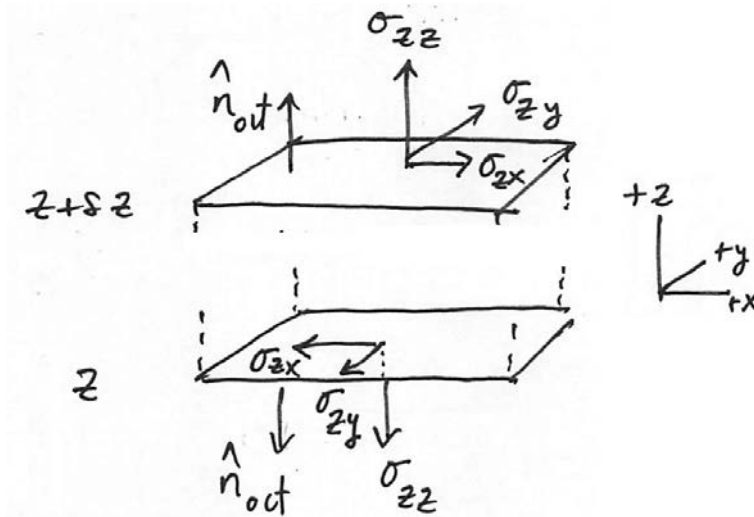Let's now consider the surface forces on our cubic fluid element.
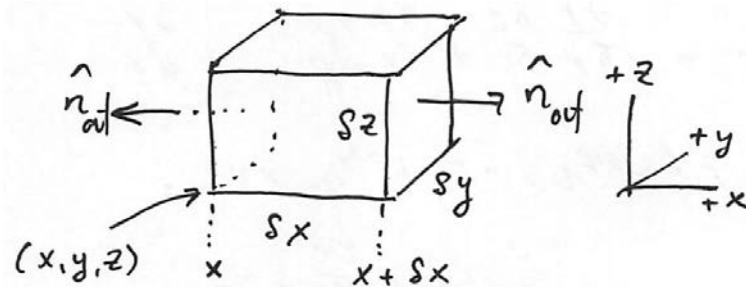
We begin by considering the forces in the x-direction. We first consider the forces acting on the two surfaces with fixed values of x and $x + \delta x$ (Figure 29.).

positive stresses
for $\hat{n}_{out} = -\hat{i}$

positive stresses
for $\hat{n}_{out} = +\hat{i}$
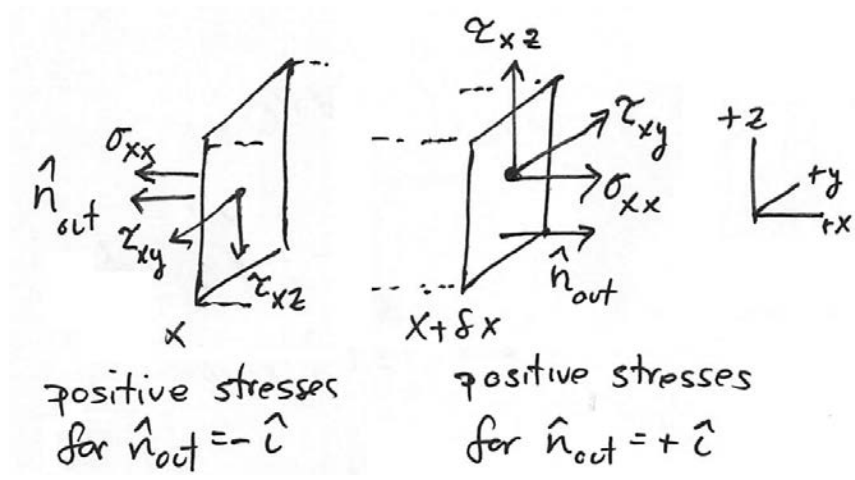
The contribution to the x-component of the force is only due to the normal stress times the area of the surface element $\delta z \delta y$, and is given by

$$(\sigma_{xx}(x+\delta x) - \sigma_{xx}(x))\delta z \delta y = \frac{\partial \sigma_{xx}}{\partial x}\delta x \delta z \delta y = \frac{\partial \sigma_{xx}}{\partial x}\delta V . \qquad (29.3.7)$$

Now let's consider the pair of top and bottom surfaces as shown in Figure 29. .

The contribution to the x-component of the force is due to the shear stresses tiems the surface area $\delta x \delta y$

$$(\sigma_{zx}(z+\delta z) - \sigma_{zx}(z))\delta x \delta y = \frac{\partial \sigma_{zx}}{\partial z}\delta z \delta x \delta y = \frac{\partial \sigma_{zx}}{\partial z}\delta V . \qquad (29.3.8)$$

The forces on the front and back facing planes are shown in the Figure 29. .

29-10

The contribution to the x-component of the force is due to the shear stresses times the surface area $\delta z \delta x$

$$(\sigma_{yx}(y+\delta y) - \sigma_{yx}(y))\delta z \delta x = \frac{\partial \sigma_{yx}}{\partial y}\delta z \delta x \delta y = \frac{\partial \sigma_{yx}}{\partial y}\delta V .$$ (29.3.9)
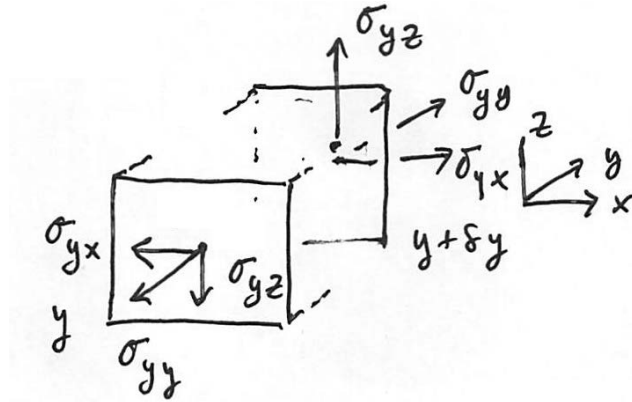
The total x-component of the force on all of the surface faces of the cubic fluid element is then

$$\delta F_{s,x} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} + \frac{\partial \sigma_{yx}}{\partial y} \right)\delta V .$$ (29.3.10)

In a similar calculation, the total y- and z-component of the force on all of the surface faces of the cubic fluid element is

$$\delta F_{s,y} = \left(\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \frac{\partial \sigma_{xy}}{\partial x} \right)\delta V ,$$ (29.3.11)

$$\delta F_{s,z} = \left(\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial x} \right)\delta V$$ (29.3.12)

29.5.2 Body Forces

In addition to surfaces forces there may be body forces acting on the entire fluid element. The gravitational force on the fluid element is an example of such a body force and is given by

$$\delta \vec{F}_g = \rho \vec{g} \delta V$$ (29.3.13)

The combined body and surface forces on the fluid element is then

$$\delta F_x = \left(\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} + \frac{\partial \sigma_{yx}}{\partial y} \right) \delta V,$$

$$\delta F_y = \left(\rho g_y + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \frac{\partial \sigma_{xy}}{\partial x} \right) \delta V, \qquad (29.3.14)$$

$$\delta F_z = \left(\rho g_z + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial x} \right) \delta V.$$

29.5.3 Non-viscous Fluids

For many fluids in which the viscosity is very small, the shearing stresses are negligible. The flow fields associated with negligible shearing stresses are said to be *non-viscous* or *inviscid*,

$$\sigma_{ij} = 0, \quad i \neq j \quad \text{(non-viscous)} \qquad (29.3.15)$$

When there are no shearing stresses the normal stresses on any fluid element are independent of direction. The pressure is defined to be the negative of the normal stress.

Recall that the normal stress is positive if the normal surface force is directed outward from the fluid element,

$$-p = \sigma_{xx} = \sigma_{yy} = \sigma_{zz} \quad \text{(non-viscous)}. \qquad (29.3.16)$$

For a non-viscous fluid the total force on the fluid element is then

$$\delta F_x = \left(\rho g_x - \frac{\partial p}{\partial x} \right) \delta V$$

$$\delta F_y = \left(\rho g_y - \frac{\partial p}{\partial y} \right) \delta V \qquad (29.3.17)$$

$$\delta F_z = \left(\rho g_z - \frac{\partial p}{\partial z} \right) \delta V.$$

The vector expression for the total force is then

$$\delta \vec{\mathbf{F}} = \left(\rho \vec{\mathbf{g}} - \vec{\nabla} p \right) \delta V . \qquad (29.3.18)$$

29.6 Euler Equations of Motion for a Non-Viscous Fluid

Consider a non-viscous flow (no shearing stresses) $\vec{\mathbf{v}}(x,y,z,t)$ with pressure $p(x,y,z,t)$, and density $\rho(x,y,z,t)$. We can now apply Newton's Second Law, $\delta \vec{\mathbf{F}} = \delta m \vec{\mathbf{a}}$ to a small

fluid element of mass $\delta m = \rho \delta V$. The force on the fluid element is given by Eq. (29.3.18), and the acceleration is given by Eq. (29.1.13). Therefore Newton's Second Law, after dividing through by the volume δV is

$$\rho \vec{\mathbf{g}} - \vec{\nabla} p = \rho \left(\frac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\nabla}) \vec{\mathbf{v}} \right). \tag{29.4.1}$$

Eq. (29.4.1) is called *Euler's Equations of Motion* for a non-viscous fluid. This is a non-linear equation that arises due to the convective derivative $\vec{\mathbf{v}} \cdot \vec{\nabla}$.

29.6.1 Steady Flow

For steady flow,

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} = \vec{\mathbf{0}} \quad \text{(steady flow)} \tag{29.4.2}$$

We can rewrite Eq. (29.4.1) using the vector identity

$$(\vec{\mathbf{v}} \cdot \vec{\nabla}) \vec{\mathbf{v}} = \frac{1}{2} \vec{\nabla} (\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}) - \vec{\mathbf{v}} \times (\vec{\nabla} \times \vec{\mathbf{v}}). \tag{29.4.3}$$

Recall that the gravitational field is the gradient of the potential function

$$\vec{\mathbf{g}} = -\vec{\nabla} \phi_g. \tag{29.4.4}$$

Then for steady flow Eq. (29.4.1) becomes

$$-\rho \vec{\nabla} \phi_g - \vec{\nabla} p = \rho \left(\frac{1}{2} \vec{\nabla} (\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}) - \vec{\mathbf{v}} \times (\vec{\nabla} \times \vec{\mathbf{v}}) \right) \quad \text{(steady non-viscous flow)}. \tag{29.4.5}$$

29.6.2 Incompressible Steady Flow

Let's now consider an incompressible fluid. Then we can rearrange Eq. (29.4.5) as

$$\rho \vec{\nabla} (\phi_g + \frac{p}{\rho} + \frac{1}{2} v^2) = \vec{\mathbf{v}} \times (\vec{\nabla} \times \vec{\mathbf{v}}) \quad \text{(incompressible steady non-viscous flow)}. \tag{29.4.6}$$

29.6.3 Irrotational Incompressible Steady Flow: Bernoulli Equation

Recall that for irrotational flow (Eq. (29.2.19), $\vec{\nabla} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$. Therefore for steady irrotational flow, the Euler Equations of Motion become

$$\rho \vec{\nabla}(\phi_g + \frac{p}{\rho} + \frac{1}{2}v^2) = \vec{0} \quad \text{(irrotational incompressible steady non-viscous flow)} . \text{(29.4.7)}$$

Define a new potential function

$$\phi \equiv \phi_g + \frac{p}{\rho} + \frac{1}{2}v^2 . \tag{29.4.8}$$

Then Eq. (29.4.7) can be rewritten after dividing through by the density ρ

$$\vec{\nabla}\phi = \vec{0} \quad \text{(irrotational incompressible steady non-viscous flow)} \tag{29.4.9}$$

This implies that

$$\phi_g + \frac{p}{\rho} + \frac{1}{2}v^2 = \phi_0 \quad \text{(irrotational incompressible steady non-viscous flow)} \tag{29.4.10}$$

.
Eq. (29.4.10) is called the *Bernouilli Equation*. Consider any two points A and B. Then the constancy of ϕ implies that

$$\phi_{g,A} + \frac{p_A}{\rho_A} + \frac{1}{2}v_A^2 = \phi_{g,B} + \frac{p_B}{\rho_B} + \frac{1}{2}v_B^2 \tag{29.4.11}$$

(irrotational incompressible steady non-viscous flow).

29.10 Navier-Stokes Equations for an Incompressible Newtonian Fluid

29.10.1 Stress and Deformation for an Incompressible Newtonian Viscous Fluid

For a viscous fluid, the normal stress on each face of the small fluid element are no longer necessary equal. When the normal stresses are related to the pressure and the rate of change of the velocity gradient by

$$\sigma_{xx} = -p + 2\mu \frac{\partial v_x}{\partial x}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v_y}{\partial y} \tag{29.4.12}$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z}$$

the fluid is called **Newtonian**, where μ is the **viscosity** and has SI units $[\mathrm{N} \cdot /(\mathrm{m}^2 \cdot (\mathrm{m/s})/\mathrm{m})] = \mathrm{N} \cdot \mathrm{s} \cdot \mathrm{m}^{-2} = \mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-1}$. If we add the three normal stresses we have that

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -3p + 2\mu\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right). \tag{29.4.13}$$

Recall that the continuity equation for the incompressible fluid element is given by

$$\vec{\nabla} \cdot \vec{\mathbf{v}} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0. \tag{29.4.14}$$

Therefore the pressure is the negative of the average value of the three normal stresses,

$$p = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \quad \text{(incompressible Newtonian Fluid)}. \tag{29.4.15}$$

The shear stresses for a Newtonian fluid are related to the rate of shearing strain (the gradient of the components of the velocity vector, $\vec{\nabla} v_x, \vec{\nabla} v_y, \vec{\nabla} v_z$ by

$$\sigma_{yx} = \sigma_{xy} = \mu(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}),$$

$$\sigma_{zy} = \sigma_{yz} = \mu(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}), \tag{29.4.16}$$

$$\sigma_{xz} = \sigma_{zx} = \mu(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}).$$

For a non-Newtonian fluid, the shear and normal components are proportional to higher order partial derivatives of the components of the velocity field.

Derive this.

29.10.2 Navier-Stokes Equations for an Incompressible Newtonian Fluid

We can now substitute Eqs. (29.4.12) and (29.4.16) into Eqs. (29.3.14) to determine the components of the force on the fluid element. We begin with the x-component of the force on the fluid element

$$\delta F_x = \left(\rho g_x - \frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 v_x}{\partial x^2} + \mu\frac{\partial^2 v_x}{\partial z^2} + \mu\frac{\partial^2 v_z}{\partial z \partial x} + \mu\frac{\partial^2 v_y}{\partial y \partial x} + \mu\frac{\partial^2 v_x}{\partial y^2}\right)\delta V, \tag{29.4.17}$$

which we can rewrite as

$$\delta F_x = \left(\rho g_x - \frac{\partial p}{\partial x} + \mu \frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) + \mu \frac{\partial^2 v_x}{\partial x^2} + \mu \frac{\partial^2 v_x}{\partial z^2} + \mu \frac{\partial^2 v_x}{\partial y^2} \right) \delta V \quad .(29.4.18)$$

Recall that the continuity equation for the incompressible fluid element is given by

$$\vec{\nabla} \cdot \vec{\mathbf{v}} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \qquad (29.4.19)$$

Therefore the x-component of the force on the fluid element

$$\delta F_x = \left(\rho g_x - \frac{\partial p}{\partial x} + \eta \frac{\partial^2 v_x}{\partial x^2} + \eta \frac{\partial^2 v_x}{\partial z^2} + \eta \frac{\partial^2 v_x}{\partial y^2} \right) \delta V \quad . \qquad (29.4.20)$$

We can calculate the y-component of the force on the fluid element in a similar fashion

$$\delta F_y = \left(\rho g_y - \frac{\partial p}{\partial y} + 2\mu \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial}{\partial z}\mu(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}) + \frac{\partial}{\partial x}\mu(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}) \right) \delta V$$

$$\delta F_y = \left(\rho g_y - \frac{\partial p}{\partial y} + \mu \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) \right) \delta V \quad . \ (29.4.21)$$

$$\delta F_y = \left(\rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) \right) \delta V$$

The z-component of the force on the fluid element is

$$\delta F_z = \left(\rho g_z + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial x} \right) \delta V$$

$$\delta F_z = \left(\rho g_z - \frac{\partial p}{\partial z} + 2\mu \frac{\partial^2 v_z}{\partial z^2} + \frac{\partial}{\partial y}\mu(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}) + \frac{\partial}{\partial x}\mu(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}) \right) \delta V$$

$$\delta F_z = \left(\rho g_z - \frac{\partial p}{\partial z} + \mu \frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) \right) \delta V \qquad . \ (29.4.22)$$

$$\delta F_z = \left(\rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) \right) \delta V$$

The vector expression for the force acting on the fluid element is then

$$\delta \vec{\mathbf{F}} = \left(\rho \vec{\mathbf{g}} - \vec{\nabla} p + \mu \nabla^2 \vec{\mathbf{v}} \right) \delta V \quad \text{(incompressible Newtonian Fluid)} . \qquad (29.4.23)$$

We now use Eq. (29.1.11) for the acceleration field in Newton's Second Law and after dividing through by the volume δV becomes

$$\rho \vec{\mathbf{g}} - \vec{\nabla} p + \mu \nabla^2 \vec{\mathbf{v}} = \rho \frac{\partial \vec{\mathbf{v}}}{\partial t} + \rho (\vec{\mathbf{v}} \cdot \vec{\nabla}) \vec{\mathbf{v}} \quad \text{(incompressible Newtonian Fluid)} . \, (29.4.24)$$

Eq. (29.4.24) is called the **Navier-Stokes equations** and applies to an incompressible Newtonian fluid. The continuity equation and the Navier-Stokes equations are four independent equations in four unknowns, p, v_x, v_y, and v_z. This is a set of non-linear equations partial differential equations that in principle are enough to solve for our four unknowns but there are only a few special cases in which there is exact analytic solutions. When we compare the Navier-Stokes equations to the Euler equations of motion for the incompressible non-viscous fluid we see that the new term due to viscosity, $\mu \nabla^2 \vec{\mathbf{v}}$, is proportional to the Laplacian of the velocity field.

29.10.2 Navier-Stokes Equation in Polar Coordinates

There are many vector flow fields that exhibit cylindrical symmetry, for example flow in a cylindrical pipe. If we choose polar coordinates (r, θ, z), (Figure 29.) then the gradient operator is given by the expression

$$\vec{\nabla} = \frac{\partial}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{\partial}{\partial z} \hat{\mathbf{k}} . \qquad (29.4.25)$$

The Laplacian operator is given by

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} . \qquad (29.4.26)$$

Example Laminar Flow in a Cylindrical Pipe

Consider a pipe of length L and radius R. A Newtonian fluid with viscosity μ undergoes steady irrotational flow. Choose cylindrical coordinate with the origin located at the center of the pipe and the positive z-direction in the direction of the flow. The only non-zero component in the velocity field is in the z-direction, thus $\vec{\mathbf{v}} = v_z(r) \hat{\mathbf{k}}$. Let's neglect the effect of the gravitational force on the flow. The Navier-Stokes equations in the positive z-direction becomes

$$-\frac{\partial p}{\partial z} + \mu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) = 0 . \qquad (29.4.27)$$

$$-\rho g\sin\theta - \frac{\partial p}{\partial r} = 0. \tag{29.4.28}$$

$$-\rho g\cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} = 0. \tag{29.4.29}$$

We begin by integrating Eq. (29.4.28)

$$-g\sin\theta \int_0^r dr' = \int_{p(0)}^{p(r,\theta)} dp'. \tag{29.4.30}$$

resulting in

$$-gr\sin\theta = p(r,\theta) - p(0). \tag{29.4.31}$$

Integrating Eq. (29.4.29) yields

$$-\rho g r \int_{\theta'=0}^{\theta'=\theta} \cos\theta'\,d\theta' = \int_{p(0)}^{p(\theta)} dp'$$
$$-\rho g r \sin\theta = p(\theta) - p(0) \tag{29.4.32}$$

Recall that $r\sin\theta = h$, therefore

$$-\rho g h = p(r,\theta) - p(0). \tag{29.4.33}$$

We can also integrate Eq. (29.4.27)

$$\int_0^r \frac{\partial p}{\partial z} r'\,dr' = \int_0^r \mu \frac{\partial}{\partial r'}\left(r'\frac{\partial v_z}{\partial r'}\right) dr'$$
$$\frac{1}{2\mu}\frac{\partial p}{\partial z}r^2 = \left(r\frac{\partial v_z}{\partial r}\right) \tag{29.4.34}$$

where we used the fact that $\partial p / \partial z$ is independent of r and so can be treated as a constant with regards to the integration. Integration again yields

$$\frac{1}{2\mu}\frac{\partial p}{\partial z}\int_0^r r'\,dr' = \int_{v_z(0)}^{v_z(r)}\left(\frac{\partial v_z}{\partial r'}\right) dr'$$
$$\frac{1}{4\mu}\frac{\partial p}{\partial z}r^2 = v_z(r) - v_z(0) \tag{29.4.35}$$

Therefore the z-component of the velocity field is

$$\frac{1}{2\mu}\frac{\partial p}{\partial z}\int_0^r r'\,dr' = \int_{v_z(0)}^{v_z(r)}\left(\frac{\partial v_z}{\partial r'}\right)dr'$$

$$v_z(r) = \frac{1}{4\mu}\frac{\partial p}{\partial z}r^2 + v_z(0)$$

(29.4.36)

At the walls of the pipe where $r = R$ there is no slippage of the fluid and so we have the boundary condition that at $r = R$, $v_z(R) = 0$. From Eq. (29.4.36) this implies that

$$\frac{1}{2\mu}\frac{\partial p}{\partial z}\int_0^r r'\,dr' = \int_{v_z(0)}^{v_z(r)}\left(\frac{\partial v_z}{\partial r'}\right)dr'$$

$$v_z(0) = -\frac{1}{4\mu}\frac{\partial p}{\partial z}R^2$$

(29.4.37)

Therefore the velocity field is

$$\vec{\mathbf{v}}(r) = v_z(r)\hat{\mathbf{k}} = \frac{1}{4\mu}\frac{\partial p}{\partial z}(r^2 - R^2)\hat{\mathbf{k}}\ .$$

(29.4.38)

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