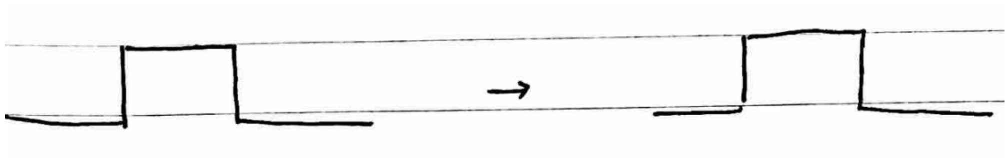
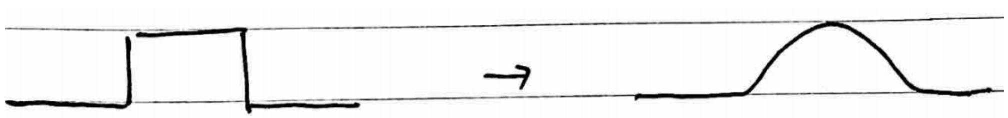
8.03 Lecture 14

In many media: $v = v(x)$. For instance: light in glass.
*Speed of wave propagation depends on wavelength λ (or ω or k).
In a non-dispersive medium, waves keep their shape:



In a dispersive medium, waves spread out:



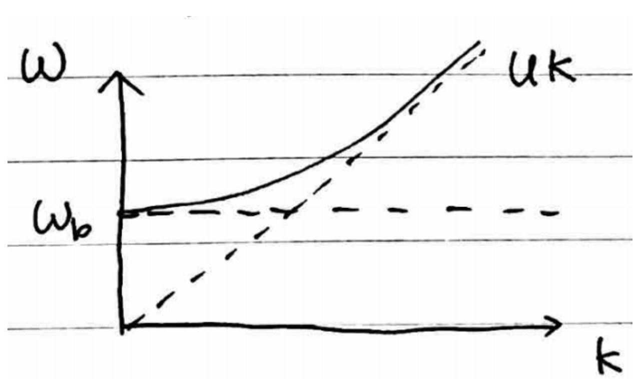
This spreading out occurs because the square wave on the left is made of many different modes, and they all travel at different speeds so the wave "disperse"
Recall our definitions of phase and group velocity:

$$v_p = \frac{\omega}{k} \qquad v_g = \frac{d\omega}{dk}$$

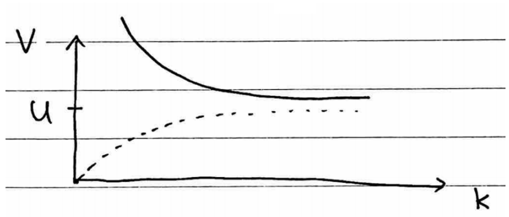
In a non-dispersive medium: $\omega = vk$ and we have:

$$v_p = v \qquad v_g = \frac{d\omega}{dk} = v = v_p$$

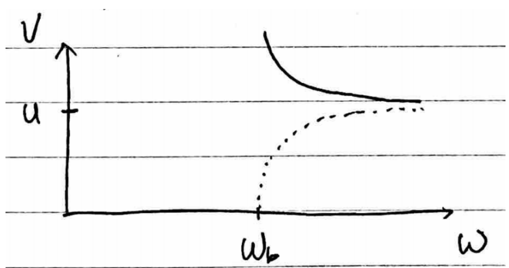
In a non-dispersive medium the phase and group velocity is the same.
Example: EM wave passing through an ionic crystal. The dispersion curve looks like:



(1.) What is the group velocity and phase velocity as a function of k?



(2.) What about velocity versus ω



(3.) What will happen to the radiation striking such a crystal if the frequency is $\omega < \omega_b$? There is no propagation or loss in this crystal $\Rightarrow$ totally reflected!

If we have a very long string:

We shake one end $\Rightarrow$ produce a progressing wave!

$$\psi(x,t) = f\left(t - \frac{x}{v}\right)$$

for non-dispersive medium. How about dispersive medium:
Waves with different frequency (or wave length) are traveling at different speeds! We need to decompose $f(t)$ into waves with fixed frequency. Then attack one by one!!!
Use the Fourier transform!

$$f(t) = \int_{-\infty}^{\infty} d\omega \underbrace{C(\omega)}_{\text{Amplitude}} \overbrace{e^{-i\omega t}}^{\text{osc. at } \omega}$$

After we decompose $f(t)$ into many harmonic oscillations with different frequency and use the dispersion relation $\omega = \omega(k)$ we have:

$$\psi(x,t) = \int_{-\infty}^{\infty} d\omega C(\omega) e^{-i\omega t + ik(\omega)x}$$

With a given ω we can solve k which is a function of ω. In the special case of the non-dispersive system:

$$k(\omega) = \frac{\omega}{v}$$

$$\begin{aligned}
\psi(x,t) &= \int_{-\infty}^{\infty} d\omega C(\omega) e^{-i(\omega t - x\omega/v)} \\
&= \int_{-\infty}^{\infty} d\omega C(\omega) e^{-i\omega(t - x/v)} \\
&= f(t - x/v)
\end{aligned}$$

Which makes sense!
How do we determine $C(\omega)$? Orthogonality:

$$\frac{1}{2} \int_{-\infty}^{\infty} e^{i(\omega - \omega')t} dt = \delta(\omega - \omega')$$

The Dirac delta function!

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

Some useful formulas:

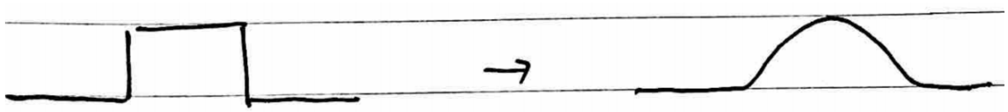
$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

$$\int_{-\infty}^{\infty} \delta(x - \alpha) f(\alpha) = f(x)$$

Now if I calculate this quantity:

$$\begin{aligned}
&\frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \left(\int_{-\infty}^{\infty} C(\omega') e^{-i\omega' t} d\omega' \right) e^{i\omega t} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} C(\omega') d\omega' \int_{-\infty}^{\infty} dt e^{i(\omega - \omega')t} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} C(\omega') \delta(\omega - \omega') d\omega' \\
&= C(\omega)
\end{aligned}$$

Now we have a problem:



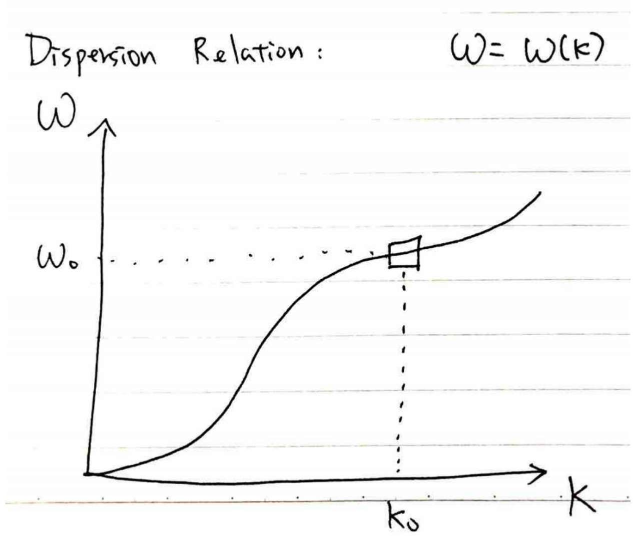
Recall from above, when we have a dispersive medium our wavepacket spreads out. How do we overcome this difficulty? A smart idea: AM radio.
Consider $f_s(t)$ is the signal we want to transmit. For instance: music around 1 KHz
Our carrier is $\cos\omega_0 t$ or $e^{i\omega_0 t}$. The frequency of AM radio is around 0.1 to 30 MHz
Instead of transmitting $f_s(t)$ directly (we know this does not work) we transmit

$$f(t) = f_s(t)\cos\omega_0 t$$

Since $f_s(t)$ is slow ($\approx$ 1 KHz) compared to $\cos\omega_0 t$ ($\approx$ 0.1 - 30 MHz) the resulting ω range with non-zero $C(\omega)$ is "narrow". This is because:

$$\cos\omega_s t\cos\omega_0 t = \frac{1}{2}[\cos(\omega_0+\omega_s)t + \cos(\omega_0-\omega_s)t]$$

Where ω_s is the "typical frequency" of the signal and ω_0 is the carrier frequency. Therefore, the range of ω with non-zero $C(\omega)$ is $\approx \omega_0-\omega_s$ to $\omega_0+\omega_s$ where $\omega_s << \omega_0$



Suppose $\omega(k)$ is slowly varying around ω_0

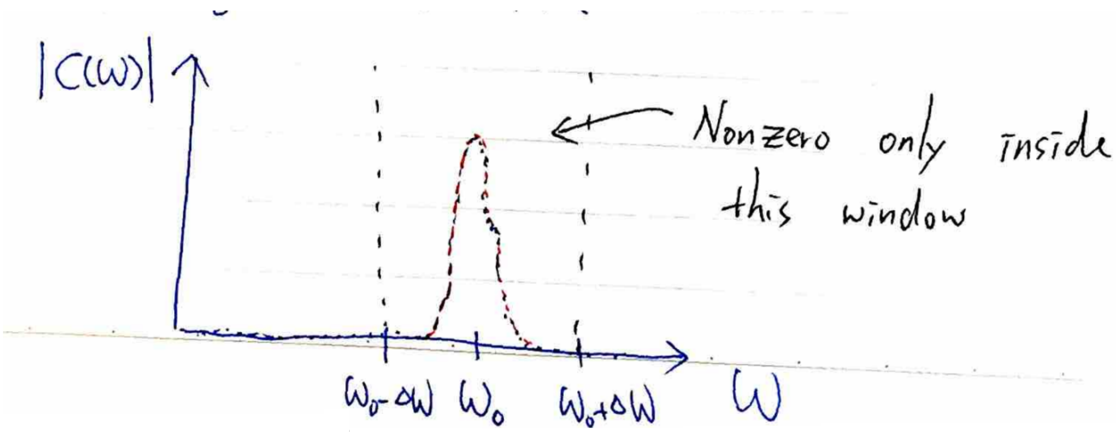
$$\omega = \omega(k) = \omega_0 + (k-k_0)\overbrace{\frac{\partial\omega}{\partial k}}^{v_g}\Big|_{k=k_0} + \cdots$$

$$\Rightarrow \ \omega \approx \omega_0 + (k - k_0)v_g$$

Where $\omega_0 \equiv \omega(k_0)$. Also note when $k \approx k_0$, $\omega \approx \omega_0$
Higher order terms are negligible in the range

$$\omega_0 - \Delta\omega < \omega < \omega_0 + \Delta\omega$$

If my $f(t)$ satisfies $C(\omega) \approx 0$ for $|\omega - \omega_0| > \Delta\omega$
I.e. we are looking at $f(t)$ with the corresponding $C(\omega)$ like:



$f_s(t)$ must be slowly varying compared to the carrier wave $e^{-i\omega_0 t}$

$$f(t) = \text{Re}[\ \underbrace{f_s(t)}_{\text{envelope}}\ \underbrace{e^{-i\omega_0 t}}_{\text{carrier}}\ ]$$

This is actually "Amplitude Modulation" or AM radio!

If $\Delta\omega << \omega_0$ (a small window with $C(\omega) \neq 0$) then higher order terms in $\omega(k)$ are negligible

$$\omega = v_g k + a \qquad a = \omega_0 - v_g k_0$$
$$k = \frac{\omega}{v_g} + b \qquad b = k_0 - \omega_0/v_g$$

Where a and b are constants. Now we want to show:

$$\psi(x,t) = \text{Re}\left[f_s(t - x/v_g)e^{-i(\omega_0 t - k_0 x)}\right]$$

Fourier transform: we can rewrite $f_s(t)$ as:

$$f_s(t) = \int_{-\infty}^{\infty} d\omega C(\omega)e^{-i\omega t}$$

Make AM radio: multiply by $e^{-i\omega_0 t}$:

$$f_s(t)e^{-i\omega_0 t} = f(t) = \int_{-\infty}^{\infty} d\omega C(\omega)e^{-i(\omega+\omega_0)t}$$
$$= \int_{-\infty}^{\infty} d\omega C(\omega - \omega_0)e^{-i\omega t}$$

Propagate to all x:

$$\psi(x,t) = \text{Re}\left[\int_{-\infty}^{\infty} d\omega C(\omega-\omega_0)e^{-i\omega t}e^{ikx}\right]$$

Recall $C(\omega)$ is only non-zero around ω_0

$$\begin{aligned}
&\approx \int_{-\infty}^{\infty} d\omega C(\omega-\omega_0)e^{-i\omega t}e^{i(\omega/v_g+b)x} \\
&= \int_{-\infty}^{\infty} d\omega C(\omega-\omega_0)e^{-i\omega(t-x/v_g)}e^{ibx} \\
&= \int_{-\infty}^{\infty} d\omega C(\omega)e^{-i(\omega+\omega_0)(t-x/v_g)}e^{ibx} \\
&= \int_{-\infty}^{\infty} d\omega C(\omega)\underbrace{e^{-i\omega(t-x/v_g)}}_{f_s(t-x/v_g)}e^{-i\omega_0 t}\underbrace{e^{i(\omega/v_g+b)x}}_{e^{ik_0x}}
\end{aligned}$$

Therefore:

$$\psi(x,t) = \text{Re}\left[f_s(t-x/v_g)e^{-i(\omega_0 t-k_0 x)}\right]$$

Where the left term ($f_s(\cdots)$) is the envelope traveling at v_g and the right term $e^{-i(\cdots)}$ is the carrier traveling at v_p What is the typical carrier frequency?
Medium frequency: 300 $KHz \leftrightarrow 3\ MHz \longrightarrow$ Skywave
High frequency: 3 $MHz \leftrightarrow 30\ MHz$
The envelope shape does not change!! (No dispersion)
Enables us to send voice, music to places which are thousands of miles away!!!