BOLESLAW WYSLOUCH: Good morning, everybody. I'm Bolek Wyslouch. I'm a teacher substitute for Professor Lee, who is now at some conference in China, and he asked me to talk to you about coupled oscillators. I understand that he introduced the concept last time. You worked through some examples.

So what we are going to do today is to basically go through one or two examples of very straightforward coupled oscillators, where I will introduce various kinds of systematic calculational techniques, how to set things up, how to prepare things for calculations. And also, we may, depending on how much time we have, start driving, have driven coupled oscillators.

And we will work on two, again, simple physical systems, one that consists of two pendula driven by forces of gravity, each of them. And then they are connected with the spring. So each of those pendula, each of those masses, will feel the effects of gravity and effects of springs at the same time, and they will talk to each other. There will be coupling between them.

So that's one physical example which we'll consider. The other physical example consists of two masses in the horizontal frictionless track connected by a set of springs. So they are driven by forces of spring. And those two systems are very similar to each other, almost identical in terms of calculations, and they exhibit the same phenomena, and I will be able to demonstrate several of the neat new things.

And this particular system is set up to introduce external driving force, which will create a new set of phenomena. And we'll talk about it today. And what I would like to stress today when we go through all those calculations is, A, how do you convert a given physical system with all the forces, et cetera, into some sort of fixed form, fixed type of notation, with which you can treat all possible coupled oscillators?

And also we will discuss various interesting-- even though the system is very simple, just two masses, a spring, a little bit of gravity on top of that, the way they behave could be extremely complex, but it can be understood in terms of very simple systematic way of looking things through normal modes and normal frequencies, so the characteristic frequencies of the system.

So let's set things up. So we'll start. This will be our workhorse. And by the way, once we understand two, we will then generalize to infinite number of oscillators, which is actually-- so this model, which consists of weights hanging under the influence of gravity plus the springs will be then used for many applications of the concepts later in this course.

So let's try to convert this physical system into a set of equations. So we have a mass, m, hanging from some sort of fixed support, another mass here, same mass for simplicity. We connect them with a string, and we know everything about this system. We know the length of each of those pendula, which is the same.

We know masses. We know spring constant of a spring connecting those two things. The spring is initially at its rest position such that when the two pendula are hanging vertically, the spring is relaxed. But if you move it away from verticality, the spring either compresses or stretches. And everything is in Earth's gravitational field, g.

We assume that this is an ideal system, highly idealized. We only consider motion with small angle approximation, only small displacement. There's no drag force assumed. The spring is ideal, et cetera, et cetera. Of course, this thing here is very far from being ideal, but hopefully basic behaviors are similar. It's approximately ideal.

To study the motion of this thing, to understand how it works, let's try to-- let's try to parameterize it, and displace it from equilibrium, and look at the forces, and try to calculate equations of motion. So we will characterize this system by two position coordinates. We will have x. We'll give this one number one. This will be number two.

And we will have x subscript 1, which in general would depend on the time. This is the position of this mass with respect to its equilibrium position. We will have x2 as a function of t. Again, this tells us everything. And the full description of the system is to know exactly what happens to x1 and x2 for all possible times.

And we will impose some initial conditions. We can come back to that later. So again, so the coordinate system is this. When we start talking about the system in principle in the case of somewhat larger angles, you have to worry about vertical positions as well. So we will introduce. So there is also a coordinate y, which we will need temporarily to set things up.

So x is, as I say, x is measured from equilibrium. Y is positioned vertically. So to calculate the equations of motion, we have to look at the forces. So let's look at what are the forces acting, for example, on this mass, the mass, which is-- if it's displaced from a vertical position.

Let's say this mass, mass 1, has moved by some distance away from thing Temporarily, let's introduce an angle here to characterize this displacement from vertical. And let's write down all the forces acting on this - force diagram acting on this mass. So there is a tension in the string or the rod. Let's call it T1.

There is a force of spring acting in a horizontal direction. This is a vector. And there is a force of gravity acting on this in the vertical direction. We can write down those forces. We know a lot about them. This one is minus mg y-hat. This one is equal to k x2 minus x1 in the x-hat direction.

So this is the force which, when the spring is displaced from equilibrium, there is a spring force, Hooke force, in the direction of -- in the usual direction. In this case, it's actually in the opposite direction. And then there is a tension the spring, which has to be calculated such that we understand the acceleration of this object.

So let's write down the equations in the x-hat direction. This is m acceleration of object number 1 in x direction is equal to minus T1 sine theta 1 plus k x2 minus x1. And in the y-hat direction, we have m y1 direction is equal to T cosine theta 1 minus mg.

At the small angle for theta 1 much, much smaller than one, we can assume, that cos theta 1 is approximately equal to 1 and sine theta 1 is equal to angle. We do the usual thing. So basically, in this approximation, and also by looking at the system, it's clear that the system does not move, and the vertical direction can be ignored. Yes?

AUDIENCE: How do you know which way [INAUDIBLE]?

BOLESLAW WYSLOUCH: Excuse me?

AUDIENCE: The [INAUDIBLE]. How do you know which way it [INAUDIBLE]?

BOLESLAW How do I know?

WYSLOUCH:

AUDIENCE: Yeah. [INAUDIBLE].

BOLESLAW WYSLOUCH: The spring force is-- well, you have to look at the mass 1. You are just looking at mass 1. So the spring is connected to mass 1. And the force of the spring on mass 1 is k times however the spring is squashed or stretched, all right?

So it knows about the existence of mass 2, but only in a sense that you have to know the position of mass 2. So we just assume that x2 is something, and we just look where the spring is. So that's why-- the force of spring depends on the difference of position x1 minus x2. So this is written here.

And in fact, interestingly, the position of the mass 1 itself is a negative sign here. So if you move mass 1, the spring force is in the right direction, minus kx. All right?

So there is no motion x1, so we can conclude from here the T cosine1 is approximately equal to 1. So T is simply equal to mg. So the tension in the spring can be assumed to be mg. We don't have to worry about it. And then we just plug in-- also the angle can be converted into position by realizing that the distance times the angle is equal to displacement, the usual geometry.

The net result is that by simplifying things, I can write down equations for acceleration in the horizontal direction for mass 1 is equal to minus mg x1 over l plus k x2 minus x1. OK?

So this is an equation of motion for mass 1 in our coupled system. And I could say most of the terms have to do with a motion of mass 1 itself. Mass 1 is its own pendulum. And mass 1 is feeling the effect of the spring force.

But because the force of the spring depends on the difference between positions, there is this coupling-- so the motion of mass 1 knows of where mass 2 is. And motion of mass 2 influences the motion of mass 1. That's how the coupling shows up.

So for most of those problems, what you do is you simply focus on the mass in question. You take all the forces, you calculate them, and then this coupling will somehow appear in the equations. So we can repeat exactly the same calculation focusing on mass 2. And then the equation which you will get will be very similar.

Let me just slightly rewrite this equation here to kind of combine all the terms which depend on the position of mass 1 with terms that depend on mass 2. So where m x-acceleration is equal to minus k plus mg over l times x1 plus k times x2.

So this is the coupling term. This is what makes those pendula coupled. All right? And then I can write almost exactly the same equation of mass 2 with the proper replacement of masses. So let me write this down in the following way-- kx1 minus k plus mg over l times x2.

So the motion of mass x1 depends on x1 itself multiplied by something with a spring term and gravitational term and depends on the position of mass 2 only through the spring. Mass 2 also is mostly driven by its own gravitational force of itself plus the spring depends on the position of x2.

But there is this coupling term that depends on position of mass 1. So both of them feel the neighbor on the other side, right? So if I keep this one steady of x2 equals 0, then basically the forces here is just the spring plus the gravity. If I move this one and keep this one at 0, the force on this spring spring and gravity.

But if this one is displaced, and I move that guy, the forces on this one are affected by the fact that number 2 changed. OK? Again, I was able to determine those coupling terms by simply looking at mass 1 itself, mass 2 itself. All right, so this is the set of two coupled equations. I have accelerations here for x1, x2, and I have positions here. It's like an oscillator of position acceleration with a constant term except that things here are a little mixed.

And the trick in this whole mathematics, and calculations, and the way we do things is how do you solve those coupled equations? OK? So what I would like to do is-- and there is multiple ways of doing that. So let me do everything. Let's write down everything in the matrix form, because it turns out that linear matrices are very useful for that. We will use them very, very-- in a very simple way.

So let's introduce to them and show vector, which consists of x1 and x2. So basically, all the position x1 and x2 are here. So we will be monitoring the change of this x2 as a function of time. We will introduce a force matrix k, which is equal to k plus mg over l minus k here, minus k here, k plus mg over l there. This is a two by two matrix.

And then we need a third matrix, mass matrix, which simply says that masses are mass of first object is m and the other one is also m, right? So these are three matrices that basically contains exactly the same information as out there. I probably need another matrix. I need an inverse matrix for mass, which basically is 1 over m, 1 over m, 0 and 0. This is a inverted matrix.

OK, and it turns out that after I introduced these matrices, this set of equations can be written simply as X, the second derivative of the vector capital X, is equal to minus m to the minus 1, this matrix, multiplying matrix k and then multiplying vectors x again.

All right? So this is exactly the same as this, just written a different way. So it's only the question of notation. So it turns out it's very convenient to use matrix calculation to do things faster. So instead of repeating writing, all the x1s, x2, et cetera, instead I just stick them into one or two element objects.

I use matrices to multiply things, and if I want to know x1 and x2, I can always go, OK, the top component of vector x, lower component of vector x gives me the solution. Simple. Right?

So let's try to use this terminology to find solutions. So the question is how do we find solutions to coupled oscillations. What is the most efficient way of finding the most general motion of a coupled system? Anybody knows? What's the first thing? Yes?

AUDIENCE: [INAUDIBLE].

BOLESLAW WYSLOUCH: Introduce what?

AUDIENCE: [INAUDIBLE] using complex notation.

BOLESLAW WYSLOUCH: Coupled?

AUDIENCE: Complex.

BOLESLAW WYSLOUCH: Complex oscillation. Yes, that's right. So all right, let's do it. But hold on. But what form of oscillation? OK, all kinds of complex numbers can write, but any particular--

AUDIENCE: [INAUDIBLE]

BOLESLAW WYSLOUCH: That's something. That's the physics answer, all right? Complex notation is a mathematical answer, how to solve a mathematical equation. But the physics answer is to find fixed frequency modes us such that the system, the complete system, oscillates at one frequency. Everybody moves together. This is so-called normal mode.

It turns out that every of the system, depending on number of dimensions, will have a certain number of frequencies, normal modes, that would-- the whole system oscillates at the same frequency, both x1 and x2, undergoing motion of the same frequency. We don't know what the frequency is. We don't know it's amplitude, et cetera. But it is the same. OK?

So this means that I can write that the whole vector x, both x1 and x2, are undergoing the same oscillatory motion. So I propose that-- so of course, we use the usual trick that anytime we have a solution in complex variables, we can always get back to real things by taking a real part. So I understand you've done this before.

So let's introduce variable z, just kind of a two-element vector, which has a complex term, a fixed frequency, plus a phase, a rhythm complex, multiplying vector A, a fixed vector A. OK? And vector A is simply has two components, A1, A2, or maybe I should write it differently.

So vector A contains information about some sort of initial conditions for position x 1 2. Anyway, these are two constant numbers. And also, we will, because we have this phase here, because we keep phase in this expression, we can assume and require that is a real number. So A is real. It's a slightly different way of doing things, but we can assume this for now, right?

So the solution which is written here-- it's some two numbers, oscillatory term, with both x1 and x2 oscillating with the same frequency, and this is our postulated solution. So we plug it into the equation, and we adjust things until it fits. So let's plug this into our matrix calculation.

And what you see here is that-- so what do we have? So this is the term, which is second time the derivative vector X. And because vector-- or vector Z really. So I have to do-- so I plug this here. So Z double dot is simply equal minus omega squared times Z.

Right? Like this. So this is a simple thing. When I plug this in here, my equation becomes an equation for A. So I have minus omega squared z-hat, which maybe I just write it immediately in terms of a complex term by times the vector A.

So I have e to i omega t plus y times A is equal to minus M to minus 1 K times e to the i omega t plus phi times vector A. OK? And this term is a proportionality constant at any given moment of time.

So it goes through the matrix multiplication. So you can just delete this. You can divide both sides. You have signs here. And then I have an equation which is a linear matrix equation, which is M minus 1 K times vector A.

And I can rewrite it a little bit again. So I can rewrite in this minus 1 K minus omega squared times unity matrix times vector A is equal to 0. So this is the equation which we need to solve to obtain the solutions to at least one normal mode, and we expect that there will be two normal modes, because we have two masses.

So now, this is-- so this is some matrix, two by two matrix, which we can know very easily how to write. Multiplying a vector gives you 0. It turns out that for this to work, there are two-- there is a criterion, which has to be satisfied, namely the determinant of the two by two matrix has to be equal to 0, because if you take the determinant on both sides, you have to have 0 on this side to be able to obtain 0 on the other side. So mathematically, the way to find out the oscillating frequency is you take a determinant of m minus 1 K minus i omega squared must be equal to 0.

So let's try to see how to calculate things. So let's write down this matrix explicitly using this and that. So let's write this down. So I take a big object like this. And so in this element here, I have to multiply this matrix times that. If I multiply this matrix, I simply divide all those effectively multiplication of m minus 1 times this matrix divides all the elements here by m. That's all there is to it. I just divide everything by m. So the first M minus 1 K is k over m plus g over l. This is minus k over m minus k over m k over m plus g over l.

So this is multiplication. This is this term here. And then I have to do minus unity matrix times omega squared. All this will do is it will subtract omega squared here. I should write this. OK? So this is in this one. Maybe it would be more clear if I move it over here.

All right, so this is the matrix that contains all the information about our system, the mass, the gravitational acceleration, the length, the spring strength, et cetera. And we assumed they oscillate with a fixed frequency. So I have to find the determinant of this matrix equal to 0.

So how do I get that? And by the way, you have a matrix, and you want to make sure that its determinant is 0. It turns out the only variable which we have to change parameters of this matrix-- you know, the spring constant and the mass this affects is given. The system has been built. It's hanging over there. I cannot change anything.

So the only parameter here, which I can change, or adjust, or find is omega square. So I will try all possible matrices of this type until I find one or two that have a determinant equal to 0. But if I find them, this would correspond to the normal frequencies. OK?

So how do I calculate the determinant of a two by two matrix? I do this by this minus this by that, right? So that of this matrix is equal to k over m minus g plus g over l minus omega squared. The two identical terms so I can put the square and then minus this minus k squared over m squared must be equal to 0.

Right? So this is the equation which we need to solve. We need to find which parameter omega sets this to 0. And then this is a pretty straightforward calculation, except if I don't have-- I'll just use this one.

OK, so let's rewrite this a little bit. So this is basically equivalent to the following equation g over l plus k over m minus omega squared must be equal either to plus or minus k over m. Right? I took a square root of both sides. If you take a square root, you have to worry about plus and minus signs, right?

So there are two solutions which corresponds to plus here. The other one corresponds to minus here. So solution number 1, which corresponds to plus sign right here, it basically says that omega squared is equal to g over l. Right? So there is one solution, one oscillation, that does not depend on the spring constant, because the spring constant cancels.

And there's a second solution which corresponds to minus, where omega squared is equal to g over l plus 2k over m. Right? Because there are two possible solutions. And this is what we have. So we have a-- so what this says is that if I set my frequency to g over l, if I set the system to oscillate to this frequency, then it will be-- I will be able to set things up such that it oscillates forever at this frequency, one fixed frequency forever.

And this is interesting. This is a frequency. It does not depend on the strength of the spring. How is it possible? Somehow spring is irrelevant for this motion. And it turns out that there is a very simple oscillation, easy to see, if basically that this is a frequency of a single pendulum. So basically, you got both pendula going together, each of them happily oscillating by themselves.

And the spring is completely irrelevant for this motion. If I cut it off, the motion will not change.

It just happens that two identical pendula are going at their own natural frequency. So the force of spring is irrelevant. Nothing happens. This is a normal mode. And it can go forever at this particular frequency. OK?

The other option is usually symmetrically. I move them away from each other. And this is the motion where, again, it's not exactly ideal small angle oscillation, but let me try again, I guess with less. So this is the situation where the spring really comes in at full force. It's being stretched maximally, because they go away from each other. So very quickly, the spring is stretched. And they go together so it's stretch from both sides.

And the whole system oscillates at the same frequency, and because of this additional force of spring, the frequency is actually higher, it's larger. It oscillates faster. All right, so that's the first step in understanding the system. We now know that there are two oscillations and two normal frequencies. And the next step to finish our understanding of the system in a mathematical way, to describe it fully, I have to know what is the shape of oscillations.

I simply showed you here so you know what to expect. But I have to be able to dig it out from the equations. And the way to dig it out is to find vector A. See, our real equation of motion is up here. This is an equation of motion. This is, I have to now find the vector A, which when you plug it in, it works-- it satisfies this equation.

So I already know what are the two possible omegas-- they can do it, but still I have to find vector A. So I have to solve two separate independent problems. One is finding vector A for this situation and then find the vector A for that situation and see if it works. So I had to plug in the whole. I had to plug it into the whole equation.

And you can show that if you set-- if you set omega squared to g over l, and you plug it into-- if you plug it into this equation, what you get is a matrix equation which looks like this-- k over m minus k over m minus k over m k over m. And this is because-- [I try to-- so if you plug omega squared here equal to g over l, then this cancels out, and this cancels out.

So you plug it in here, and you get this very simple, very simple matrix that has k over m terms. So the question is what sort of thing can you put here to get 0. What kind of vector you can plug into those two places such that the matrix times vector will end up with 0?

One example is that basically amplitude is the same. Both of them move together. So you plug 1 here and 1 here. Right? So this is a good solution. And every other solution is a linear multiplication of this one for this frequency, right? There is k over m times 1 minus k over m gives you 0. So this is a good solution for-- so this is solution number 1.

What about this thing here? If I plug this omega squared into this matrix, it's g over l plus 2k over m. If I plug it in here, then this matrix is way more complicated. It will actually look very similar, but with important differences.

So this one will look minus k over m minus k over m minus k over m minus k over m. OK? And then again, for this second possible normal frequency, I have to find the vector A, which corresponds to that frequency motion. And it turns out that they are the same, but the sign changes.

So one possible solution is 1 and minus 1. If I plug in 1 minus 1, then this matrix times the vector gives you automatically 0. So this is the second possible normal mode. All right? So this is a systematic way to solve equations. You plug in all the information you know about the system into a two by two matrix.

And then you calculate the normal mode. And then you calculate a shape of a normal mode. Is that clear? Any questions at this time? Right? So in principle, we know, now, at the end of the day, I still want to know how much 1 moves, how much 2 moves. So we have to put it all together.

We have identified the frequency and the kind of, in the matrix notation, shape of the node. But of course, the final solution is a linear superposition of all possible normal modes with described position of mass 1, position of mass 2, et cetera. So let's do a little bit of-- so maybe graphically I can write down that this is the-- this is the oscillation that corresponds to this type of mode, to those two masses move together.

And this is oscillation that corresponds to the mode where masses move in opposite directions. At any moment of time, in this normal mode, at any moment of time, wherever mass 1 is, mass 2 is minus the distance away from its own equilibrium. So if this one is plus 1 centimeter here, the other one is minus 1 centimeter. This one is minus 5. This one is plus 5 and so on.

Whereas in this mode, both of them move together. All right? So let's try to go back to the-- you can get rid of this one. Let's try to go back, and now with this knowledge, let's write down x1 and x2 for positions of the two masses.

So x1-- so basically, the x will have to be-- I used z there. So x will be real of vector z. So I take my complex numbers and take a real of them. So from an exponent, I will end up with a cosine appropriately and so on. And then I will use the [INAUDIBLE].

So this is real part of e to the i omega plus phi where omega is one of the two possibilities. Omega t plus phi times vector A, which we've identified here, and times some additional-- these are those vectors A this is one possible amplitude of notation. But in general, it can be anything. You can multiply. You can have small oscillations, large oscillations. So there is some overall amplitude. But the shape always has to be simple. They either go together, or they go opposite. So to make it more general, I have to give some multiplicative factor there.

So if I do everything, I end up with x, the mode 1 will in general have some sort of overall constant C, cosine omega 1 t plus phi 1 times the vector 1, 1. This will be for x1. This will be for x2. And the mode number 2 will be C2 cosine omega 2 times t plus phi 2 times 1 minus 1.

All right? So let's see what things are adjustable and what things are fixed. So the omega 1 and omega 2 are fixed given by the construction of the two coupled oscillators. This shape, 1 and 1, and 1 minus 1 is fixed, because these are the shape of normal modes, which corresponds to those frequencies.

So we have only four constants-- overall amplitude c1 for normal mode 1. Overall amplitude c2 for normal mode 2 and then the relative phase of those two normal modes. And the superposition of x1 plus x2 gives you the most general combination of possible motion. So if I write this down now in terms of position of number 1 and number 2, so I have a position of x1 as a function of time. In general, it will look like this. It will be some sort of constant alpha, cosine omega 1 t plus phi plus constant beta cosine omega 2 times t plus phi 2 plus phi 1.

So mass number 1, this is position of mass 1, will in general be a superposition of the two possible oscillations. The position of mass 2 will be very similar, but there will be a very important difference between the alpha cosine omega 1 t plus phi 1 minus beta cosine omega 2t plus phi 2.

This is very important to understand exactly how this equation came about. You see, this is the influence of the symmetric mode, where the two things are together. So they are multiplied by alpha, some sort of arbitrary constant, but with exactly the same sign. And this is the part which corresponds to a second mode, which is with different frequencies. And there is an opposite sign between this amplitude and that amplitude.

So you have only four coefficients-- alpha, beta, phi 1, and phi 2, which are determined, which need initial conditions. So any arbitrary mode-- this is the most general motion of the two coupled oscillator systems. And to describe it in specifically-- defined for a specific configuration, you will have to determine the values of alphas and phis. OK?

So what I want to do is I want to write down a specific motion for the following situation. So I keep position of x1 at 0. It's not moving, so the velocity is 0. I displaced this one by a small positive amount. So the position of number 2 at t equals 0 is different than 0-- some displacement x0 or something.

And its velocity is 0. And then I let it go. Again, this is not the ideal decoupled oscillator, right? OK, and then you see the things start moving. Let me try to show it again, because it's not exactly here, so this one will be going on. So let's say this one is running, and then I let this one go.

And what you see here is that this one is moving, and then that starts to move. This one stops. That starts moving. It starts being complicated, right? It's kind of complicated motion. But whatever this motion is, we know that it's simply those cosines which are kind of adding up to give you this impression of rather a complicated motion, right?

So again, I let this one out. I let it go. This might be 0. So this one slows down. This starts going. And this one then slows down. The other one starts going. They kind of talk to each other. And it's this combination of cosines.

All right, so let's try to write to simplify this for a specific case of specific initial conditions. So I said x1 equals 0, to equal 0 x1 velocity at 0 is equal to 0. So those ones are not moving. X2 at 0 is equal to some sort of x0 and x2 velocity at 0 is equal to 0. So this one is displaced. They are all stationary. This one is at position 0.

If I plug this in, it turns out without lots of details that what you will get to is that alpha will be equal to x0 divided by 2. Beta will be equal to minus x0 divided by 2. And phi 1 will be equal to phi 2 equal to 0. You can check. If you plug it into those equations, if you plug t equals 0, phi is equal to 0, et cetera, you will see that it works.

So you can write down the specific case of x1 of t to be x0 over 2 cosine omega1 t minus cosine omega2 t. It's because beta has a negative sign. And x2 of t will be equal to x0 over 2 cosine omega1 t plus cosine omega2 t. OK?

So each of those objects effectively feels the effects of omega 1 and omega 2, but in a slightly different way. That's why their relative motions are different. So what I will do now is I will show you an animation. Hopefully, it works. And we will have time-- since on the computer, you can make things perfect. Let's do it. So I'll have-- running a MatLab simulation. Let's see how it goes.

Large. So what is going on here is the following. I took some initial conditions. I'm not sure if it's exactly the same. This was for the course that I taught some time ago. What you see here is the following-- you have the green is the normal mode, number 1. The magenta is normal mode number 2. And blue and the red are the actual pendula.

All right? And the motion of blue and red is simply a linear sum of the two. And what you see here is-- and then I plot the position of the blue and red in color, the function of time. So you see this-- the fact that let's say red is now stopped, and the blue is at maximum. And now, the red is picking up.

And now the blue stopped, and the red is going full swing, et cetera. And this is exactly what-- this is the computer simulation that shows you that one of them is going up, the other one down, et cetera. And this is for the certain combination of initial conditions. I could go change initial conditions in my program and have a different behavior.

But whatever happens, I would be able to-- it will always be a combination of the two motions. Now, is there a way to disable one of the normal modes? How would you disable one of the normal modes? Is there a quick way to set things up such that the second normal mode, whichever you choose, doesn't show up in their equations at all?

AUDIENCE: You said [INAUDIBLE].

BOLESLAW WYSLOUCH: Hmm?

AUDIENCE: [INAUDIBLE]

BOLESLAW Yeah, so what you do, is you just change the initial conditions. So you set it up at T equal to 0.

WYSLOUCH: I have initial conditions that basically favor or demand that only in this general equation either alpha or beta is equal to 0.

So for example, one possibility is I move both of them at the same distance, and I just let them go like this such that the spring is irrelevant, right? How would I do it in my program? I don't know. I can, for example-- I can, for example, set one of the initial conditions to-- this is still running. The old one is still running. So this is the moment.

So what I did is I just changed the initial condition. And you see, this is the type of motion where one of the modes has stopped, just you switched it off, and the other one is going on, and then, of course, the total motion is equal to that. And both of them happily go with a constant amplitude. There is no shifting of energy from one to another.

So you can have all kinds of motions by simply adjusting initial conditions. And those motions can be done a very different way. So do you know-- so this is how we can have different shape of motion, depending on the initial condition. Is there another way for me to change the way this system behaves?

Let's say I take-- I have exactly this system, and I want to change, for example, the frequency of oscillations. How will I do it? It could be a very expensive proposition, yes?

AUDIENCE: Drive it?

BOLESLAW WYSLOUCH: Yes, but I don't want to drive it yet. I just want to have it free oscillation. Yes?

AUDIENCE: [INAUDIBLE]

BOLESLAW WYSLOUCH: Yeah, I could come and scratch it away a little bit. And yes, the equations depends on the mass. But I don't want to touch. I want to just have this thing. I don't want to make any physical modification to the system. However, I can move it into different places, any place you can think of where I could really modify the solution. Yeah?

AUDIENCE: To the moon.

BOLESLAW WYSLOUCH: To the moon, exactly. I could put it with me some spaceship, and go to a place where the gravity is different, right? Why not?

So what would happen? So if gravity changes, then basically what will happen is both this term and that term will change. The spring will remained the same. The mass will remain the same. So the relative magnitude of omega 1 and omega 2 will change. OK?

So let's say, in fact, do I have it in this one here? Yes. So let's say I do again. So this is what I had before, right? So this is the one here which is operating here on earth, and I let it go. I displaced it by a certain distance. Let's say 1 millimeter, and that's how it's gone.

So now, let's take it to, for example, Jupiter. So what do you think will happen when we go to Jupiter. Jupiter, g, is much larger. OK? So what would happen to those? So the frequency would be larger. Things will be faster, right? That's the higher frequency.

But also the difference between two frequencies will be smaller. And what happens when the difference in frequency is smaller? You saw that there's the fact that the energy was moving from one to the other. The thing would take-- so one of them was oscillating, the other one is stationary, then the other one would pick up, et cetera. Do you think this transfer of energy will be faster or slower? Two omegas closer to each other. Any guesses?

AUDIENCE: Smaller.

BOLESLAW WYSLOUCH: Take kind of longer. Let's see what happens, right? So we go on the rocket, and nowadays, you don't have to go to the rocket. Just remove one comment. And I went from about 10 meters per square second to 25 meters per square second, and this is what is happening. Look at this.

So first of all, this identical system-- everything at the same time. It's the same. And so you see that oscillations are much faster. So a number of amplitude changes per second is larger. But it takes much longer for the energy. So the red one is now stopping. It's now slowly coming up.

So because the two frequencies are closer to each other, they stay-- it takes longer for them to shift from one to the other. OK? So we are done at Jupiter. Let's now go to the Moon, which has much lower gravitational acceleration. Let's see what happens.

Again by logical argument-- if something-- so the smaller gravitation accelerations means that the frequency is now lower. So the pendula will move slower. However, the difference between frequency will be larger, because the spring is still the same strength. So it turns out that even though everything is slower, but the energy transfer will actually be faster. So let's try to see what happens on the Moon. It's OK.

It's a little bit not completely clear what's going on, but you see, actually the motion is kind of a little strange. Look at the red one. The red one is stopping. Then it's going halfway out. It looks kind of messy, doesn't it? And so it doesn't show up here very well, because the parameters have changed so much that I have-- I have those fixed pictures which are-- just a second. I'll show you.

So this is the picture on the-- some sort of stationary picture on the Earth. I saw one of them up, the other one-- you see them shift from one to the other. And you can see kind of the frequency of how the energy shifts from one to the other. And also you can see the frequency going up and down for the same exact conditions.

This is now, just a moment, this is a Jupiter. So Jupiter, you see that the frequency itself it's much higher. And the energy transfer between the two things takes longer. And on the Moon however, the oscillations actually look really weird. This is an example of one of them.

It's kind of, you know, the two frequencies are so far away, and it's really not even a nice oscillatory motion. It's some sort of-- it's much less obvious that this is a superposition of two cosines, because they kind of are exactly out of phase. So the motion is kind of complete.

Anyway, so this is-- actually, so the lesson is that the exact shape, the exact motion, we know that can always be decomposed into simple motions. If you put them together, things may get really interesting and complicated, depending on what sort of frequencies we are running and what sort of-- what sort of initial conditions we have.

All right? Yes? Any questions? Yes?

AUDIENCE: It's talking about the center mass of the system or just one of the two --?

BOLESLAW WYSLOUCH: This one, I think, this one is just one of them. Actually, the one-- on the difference-- it normally doesn't matter. What matters this is the frequency and how these move to the other. OK? Let's just forget about it. Just keep it.

So let me now talk about this thing, which is called beat phenomenon, because when you look at the motion of one of those objects, or the difference between them or whatever, there's something kind of interesting which can be extracted for those equations.

Let's look at these equations here. Let's look at mass 1. This is mass 1 and mass 2. So I can rewrite those solutions a little bit different. And so what I want to do is I want to-- you see, this is a difference of two cosines. This is a sum of two cosines.

There are lots of neat trigonometrical identities which we can use. So we just-- we do zero physics here. We just rewrite the trigonometrical formulas. So I do exactly this, but I rewrite it. I use, for example, some of-- you have cosine alpha plus cosine beta is equal to-- two cosine-- is equal to two cosine alpha plus beta divided by 2 multiplied by cosine alpha minus beta divided by 2.

Right? That's the trigonometric identity. Right? So let's just use this to write this down and what you get is x1-- x1 of t is equal to minus x0 sine of omega 1 plus omega 2 divided by 2 times sine omega 1 minus omega 2 divided by 2 times t. And x2 t is equal to x0, some amplitude cosine omega 1 plus omega 2 divided by 2 cosine omega 1 minus omega 2 divided by t.

So again, we did zero physics here. We just rewrote the simple trigonometric equations. But what you see is something interesting here. So there is-- we have those two frequencies which are playing a role. And for example, at Jupiter, those two frequencies are actually very close to each other, because everything is dominated by the gravity, and we have a very weak spring.

So the omega 1 and omega 2 actually are very close to each other. So this thing, this term here, kind of goes omega 1 plus omega 2 divided by 2 is like omega, right? 100 plus 105 divided by 2 is about 100. Whereas this one here carries information about the difference of frequencies-- 100, 102, the difference is 2, which is very small.

So how would this look like? So if you make a plot under some conditions, you can, let's say, so the two frequencies are close to each other. So if omega 1 is close to omega 2-- for example, omega 1 is 0.9 times omega 2, right? This is roughly what we have on Earth in case of our system here.

Then omega 1 plus omega 2 divided 2 would be about 0.95 omega 1, omega 2, I think, which is approximately equal to omega 2 or omega 1 and omega 1 minus omega 2 divided by 2 will be about minus 0.05 times omega 2-- much, much smaller than that.

So we have-- so this term here-- it basically oscillates at the frequency of omega, of the frequency of the individual pendulum. And the other term is much, much smaller. How does this look? Well, it turns out that if you make a sketch of this, if you do signs, for example, it looks like this. OK?

So there are in fact two-- when you look at this picture, you can see two frequencies. One which is clear the oscillation of the-- high-frequency oscillation of things moving up and down. But there's also this kind of overarching frequency of much smaller frequency, and this is what corresponds to a difference of two things.

So in a sense, if you look at this formula here, you have oscillation, which is happening very quickly with a typical oscillation of the system. But this is like a modulation of the amplitude. So the amplitude of the signal is changing. And this is what you see here. This is exactly the picture out there. So the system oscillates.

So one of those pendula, either of them, is moving fast. But it's going faster. It's amplitude is larger, and after some time, it slows down to 0. It goes higher and slows down to 0. And you've seen this. We can do it again here that both of them oscillate at roughly the same frequency, but their individual amplitudes are changing.

And this transmission of-- you know, one of them moving full blast, the other one moving full blast. There's this kind of frequency of energy moving from one to the other, which is something called beat. This a beat system, beat phenomenon somehow that energy is moving from one place to another one.

And we can have some demonstration of how this happens. So we see this here. We see it on the pendula. We saw it on the computer simulation. But now what we are going to do is we're going to try to hear it, right? So this is a demonstration which maybe it works, maybe not. So let me-- it will work, OK?

So let me explain what we have. So we have two speakers. And they basically go on very, very similar frequencies, all right? So they both work at similar frequencies. And so when I switched on, you should hear-- hear the sound.

[HUM SOUND]

OK? So this is the frequency. I believe it's just one of them is working, and you know, this is just one pendulum that is going on that given frequency, right? Then I will switch a second loudspeaker.

[HUM SOUND]

Can you hear this kind of-- wiggle? We'll change the frequency a little. This is another frequency of the original sound. And it's kind of the loudness of the sound overall is changing. All right? This is faster. This is kind of extra, extra sound which you hear is the difference of mainly the frequency is not stable here, so I'll change it.

Right? So this is, again, this is a single one, perfectly constant frequency, no change in amplitude, no change in loudness. Put them together, right? That's what they do. So if you have two, and I can adjust the frequency, and the frequency is close, then this frequency of changing is very slow. So you can actually hear it.

Let me switch it off. So this is the effect of beats. I can maybe show you another simulation of this works. Let's See. This one is oops, just a second. Let's see what it is. OK, so this is just a single frequency. OK, again, I plot some pendulum.

Then I can plot-- sorry, no this one is this. I can-- this one. OK, we'll just plot it here. Maybe we can see. So there's a red one, and there's a blue one. And I plot two plots independently on top of each other. So they have an amplitude of 1. And clearly, you see that they have a different frequency.

So the red one is going with some frequency. The blue one is going with some other frequency. Sometimes they agree. Sometimes they do not agree, right? And the places where they meet-- they are on top of each other. This is where when you add them up together, this is where they will be large. In the places where they're out of phase, they will cancel each other.

So if you take two of those together, same amplitude, just slightly different frequency, and you simply make a linear-- superposition of the two, you will get exactly the beating effect. So I just took two of those pictures before I added them together and got exactly that. You have a maximum, minima, et cetera. And you see this overall beat frequency, and the carrier, it's called carrier frequency.

And this is something that, again, happens very often. There's another demonstration here. I have two tuning forks, and they are very similar frequency. So first, I will show you that they are coupled. They are coupled because I gave this guy some initial condition.

It's going. Then I stop it. But there's still sound, because the second one picked up some energy, and it took off. Of course, you don't see them. So basically, what I'm saying is that I [TONE] give this energy. This one is completely stationary. Now energy is slowly moving to the other one. I stop this guy, and this guy is still going.

So the energy is being transferred by this air oscillating here. The coupling goes through the air to the sound here, right? And they have very similar frequency. So they are nicely coupled. But what we can also do-- we can [TONE]. Right? So they're both going. Do you hear the beats?

[TONE]

Not really. In fact, if they would have exactly identical frequency, right? If they will be perfectly the same, then the difference would be 0, and there will be no beats at all. The period of beats will be infinitely long, so it will take forever for us to hear anything. So what we can do-- we can break one of them. We can add some sort of weight.

Some are here. There's some magic place where it works best. So what I would do is I will break this one. I will modify its frequency. That's another way to modify. I don't have to go to Jupiter to modify it, because this one is just a little mass here, right?

[TONE]

Ah, cool.

AUDIENCE: Is that [INAUDIBLE]?

BOLESLAW WYSLOUCH: Really, this is actually a huge effect.

[TONE]

You can clearly see that they are going up and down, up and down, because the frequency is slightly different. So now, this thing is probably-- I know it's a period, a fraction of a second, right? Yes?

AUDIENCE: Should both of those sine and cosines have Ts in their arguments?

BOLESLAW WYSLOUCH: Of course always. They are both time dependent, yeah. This is the fast thing, and this is this time-dependent modulation, yeah. All right, so where are my notes?

So this is the-- this is how the-- so we were able to set up the system, put in some of the matrix equation, kind of solved it, found two frequencies, et cetera. There is one more-- one additional trick, which you can do to describe the motion of a coupled pendula.

And that is, in a sense, force mathematically, force the normal modes from sort of early on, to instead of, so far, when we talked about pendula, we describe their motion in terms of motion of number 1, motion of number 2. It turns out we can rewrite the equation into some sort of new variables, where, so-called normal coordinates, where you'll simultaneously describe both of them and then kind of mix them together to have a new formula, just rewrite the equation in terms of new variables.

So you do change of variables. So instead of keeping track of x1 and x2 independently, you define something which I called u1, which is simply x1 plus x2, and I define u2, which is x1 minus x2. So instead of talking about x1 and x2 independently, I have a sum of them and difference.

Why not? Right? Two variables. I can always go back and get x1 and x2 if I want to. So if one tells me that u1 is 1 centimeter and u2 2 centimeters, I can always go and get x1 and x2 if I want to, right? So I can do it. And it turns out that if I plot those variables in, in other words, I take the original equations, which I conveniently erased and make a sum or difference, it turns out that this coupling kind of separates.

So I will end up having two separate equations for this one. So in general, the equation of motion would be-- would look like, so let's say I can write down m x1 plus x2 is equal to minus m g over l times x1 plus x2.

OK, this is when I add two equations. And the other equation when I subtract them-- minus x2 is equal to minus mg over l plus 2k x1 minus x2. I think that's what is coming out. So if I add and subtract the two original equations of motion, which I don't know if I have them somewhere, and you can look back, then you end up having those crossed terms drop out. And you have one, which has only this coefficient, the other one which has that coefficient.

And this immediately-- and it looks-- if I now write it in terms of normal coordinates, then I have that m u1 double dot is equal to simply minus mg over l, u1, and m u2 double dot is equal to minus mg over l plus 2k times u2. And if you look at those two equations, it turns out that they are not coupled. Each of them is a question of a one-dimensional harmonic oscillator. The first part one only depends on u1. The second one only depends on u2.

And you can see the oscillating frequency with your own eyes. So no, the determinants needed no matrices, no nothing. We just added and subtracted the two equations, and things magically separated. All right? So sometimes, especially in case of very simple and symmetric systems, if you introduce new variables, you can simplify your life tremendously, and these are called normal variables, normal coordinates.

And it turns out that you can always do that. So you can always have a linear combination of parameters for arbitrary size coupled oscillators system where you combine different coordinates, and you basically force the system to behave in a way in which it induces the single oscillation, single frequency.

So this is, again, a very powerful trick, but usually for most cases, you can do that only after you have solved it, after you've found out normal modes, et cetera. So after you know your normal mode, then you can say, ha, ha, I can I can introduce normal variables and make things simpler.

But at the end of the day for complicated systems that work is the same. But for simple systems like this one where there is a good symmetry, you can do it. Anyway, so I think we are done for today. And on Tuesday, we'll continue with forced oscillators. All right? Thank you.