YEN-JIE LEE: And welcome back, everybody, to this fun class, 8.03. Let's get started. So the first thing which we will do is to review a bit what we have learned last time. And then we'll go to the next level to study coupled oscillators. OK. Last time, we had learned a lot on damped driven oscillators. So as far as the course we've been going, actually, we only study a single object, and then we introduce more and more force acting on this object. We introduce damping force, we introduce a driving force last time. And we see that the system becomes more and more difficult to understand because of the added component. But after the class last time, I hope I convinced you that we can understand driven oscillators.

And there are two very important things we learned last time. The first one is the transient behavior, which is actually a superposition of the homogeneous solution and the steady state solution. OK. One very good news is that if you are patient enough, you shake the system continuously, and if you wait long enough, then the homogeneous solution contribution goes away. And what is actually left over is the steady state solution, which is actually much simpler than what we saw beforehand. It's actually just harmonic oscillation at driving frequency.

Also, I hope that we also have learned a very interesting phenomenon, which is resonance. When the driving frequency is close to the natural frequency of the system, then the system apparently likes it. Then it would respond with larger amplitude and oscillating up and down at driving frequency. So that, we call it resonance.

This is the equation of motion, which we have learned last time. You can see is theta double dot plus Gamma theta dot, is a contribution from the drag force, and omega 0 squared theta is the contribution from the so-called spring force. And finally, that is equal to f0 cosine omega d t, the driving force. And as we mentioned in the beginning, if we prepare this system and under-damp the situation, then the full solution is a superposition of the steady state solution, which is the left-hand side, the red thing I'm pointing to, this steady state solution.

There's no free parameter in the steady state solution. So A, the amplitude, is determined by omega d. Delta, which is the phase is also determined by omega d. There's no free parameter. OK. And in order to make the solution a full solution, we actually have to add in this homogeneous solution back into this again. And basically, you have B and alpha, those are the free parameters, which can be determined by the given initial conditions. OK.

So if we go ahead and plot some of the examples as a function of time, so the y-axis is actually the amplitude. And the x-axis is time. And what is actually plotted here is a combination or the superposition of the steady state solution and the homogeneous solution. And you can see that the individual components are also shown in this slide. You can see the red thing oscillating up and down harmonically, is steady state solution contribution. And also, you have the blue curve, which is decaying away as a function of time. And you can see that if you add these two curves together, you get something rather complicated. You will get some kind of motion like, do, do, do, do. Then the homogeneous solution actually dies out. Then what is actually left over is just the steady state solution, harmonic oscillation.

And in this case, omega d is actually 10 times larger than the natural frequency. And there's another example which is also very interesting. It's that if I make the omega d closer to omega 0-- OK, in this case it's actually omega d equal to 2 times omega 0, then you can produce some kind of a motion, which is like this. So you have the oscillation. And they stayed there for a while, then goes back, and oscillates down, and stay there then goes back. OK, as you can see on there. The homogeneous solution part and steady state solution part work together and produce this kind of strange behavior. OK. And that's just another example. And if you wait long enough, again what is actually left over is the steady state solution.

OK. So what are we going to do today? So today, we are going to investigate what will happen if we try to put together multiple objects and also allow them to talk to each other. OK. So if we have two objects, and they don't talk to each other, then they are still like a single object. They are still like simple harmonic motion on their own. But if you allow them to talk to each other, this is the so-called coupled oscillator, then interesting thing happen. So in general, coupled systems are super, super complicated. OK.

So let me give you one example here. This is actually two pendulums that are a coupled to each other, they are actually connected to each other, one pendulum, the second one is here, OK. And for example, I can actually give it an initial velocity and see what is going to happen.

You can see that the resulting motion-- OK. Remember, we are just talking about two pendulums that are connected to each other. The resulting motion is super complicated. This is one of my favorite demonstrations. You can actually stare at this machine the whole time. And you can see that, huh, sometimes it does this rotation. Sometimes it doesn't do that. And it's almost like a living creature.

So we are going to solve this system. No, probably not, he knows.

[LAUGHTER]

But as I mentioned before, you can always write down the equation of motion. And you can solve it by computer. Maybe some of the course 6 people can actually try and write the program to solve this thing and to simulate what is going to happen. So let's take a look at this complicated motion again.

So you can see that the good news is that there are only two objects. And you can see-- look at the green, sorry, the orange dot. The orange dot is always moving along a semi circle. But if you focus on the yellow dot, the yellow is doing all kinds of different things. It's very hard to predict what is going to happen.

So what I want to say is, those are interesting examples of coupled systems. But they are actually far more complicated than what we thought, because they are not smooth oscillation around equilibrium position. So you can see that now if I stop this machine and just perturb it slightly, giving it a small angle displacement, then you can see that the motion is much more easier to understand. You see. You even get one of these questions in your p-set. OK, that's good news.

So our job today is to understand what is going to happen to those coupled oscillators. Let me give you a few examples before we start to work on a specific question. The second example I would like to show you is a saw and you actually connect it to two - actually a ruler, a metal ruler, which is connected to two massive objects. Now I can actually give it the initial velocity and see what happens. And you can see that they do talk to each other through this ruler, this metal ruler.

Can you see? I hope you can see. It's a bit small. But it's really interesting that you can see-- originally, I just introduced some displacement in the left-hand side mass. And the left-had side thing start to move or so after a while.

There are two more examples which I would like to give you, introduce to you. There are two kinds of pendulums, which I prepared here. The first one is there are two pendulums that are connected to each other by a spring. And if I try to introduce displacement, I move both masses slightly and see what is going to happen. And we see that the motion is still complicated. Although, if you stare at one objects, it looks more like harmonic oscillation, but not quite. For example, this guy is slowing down, and this is actually moving faster. And now the right hand side guy is actually moving faster. Motion seems to be complicated.

Also, you can look at this one. Those are the two pendulums. They are connected to each other by this rod here. And of course, you can displace the mass from the equilibrium position. I'm not going to hit-- not hitting each other. So you can displace the masses from each other. And you can see that they do complicated things as a function the time.

How are we going to understand this? And I hope that by the end of this lecture you are convinced that you can as you solve this really easy, following a fixed procedure.

In those examples, we have two objects that are connected to each other. And therefore, they talk to each other and produce coupled motion. Those are a couple oscillator examples. There's another very interesting example, which is called Wilberforce pendulum. So this is actually a pendulum. You can rotate like this. And it can also move up and down. It's connected to a spring.

The interesting thing is that if I just start with some rotation, you can see that it starts to also oscillate up and down. You see? So initially I just introduced a rotation. Now it's actually fully rotating. And now it starts to move up and down. And you can see that the energy stored in the pendulum is going back and forth between the gravitational potential, between the potential of the spring, and also between the kinetic energy of up and down motion and the rotation. They're actually doing all those transitions all the time.

So you can see-- so initially it's just rotating. And then it starts to move up and down. And this one is also very similar. But now the mass is much more displaced. And if I try to rotate this system without introducing a horizontal direction displacement, it still does is up and down motion, like a simple spring mass system.

So what causes this kind of motion? That is because when we move this pendulum up and down, we also slightly unwind the spring. That can generate some kind of torque to this mass and produce rotational behavior.

And you can see that this is just involving one single object. But there's a coupling between the rotation and the horizontal direction motion. So that's also special kind of coupled oscillator.

So after all this, before we get started, I would like to say that what we are going to do is to assume all those things are ideal, without them being forced, without a driving force. We may introduce that later in the class. But for simplicity, we'll just stay with this idea case, before the mass becomes super complicated to solve it in front of you. And also, we can see that all those complicated motion are just illusion. Actually, the reality is that all of those are just superposition of harmonic motions. You will see that by the end of this class. So that is really amazing.

OK. Let's immediately get started. So let's take a look at this system together and see if we can actually figure out the motion of this system together. So I have a system with three little masses. So there are three little masses in this system. They are connected to each other by spring. Those springs are highly idealized, the springs. And they have spring constant k. And the natural length's l0. And they are placed on Earth. And I carefully design the lab so that there's no friction between the desk and all those little masses.

So once you get started and look at this system, you can imagine that there can be all kinds of different complicated motions. You can actually, for example, just move this mass and put the other two on hold. And they can oscillate like crazy. They can do very similar kind of motion. There are many, many possibilities. But if you stare at this system long enough, you will be able to identify special kinds of motion which are easier to understand.

So what I would like to introduce to you is a special kind of motion which you can identify from the symmetry of this system. That is your so-called normal mode.

So what is a normal mode, a special kind of motion we are trying to identify? That is actually the kind of motion which every part of the system is oscillating at the same frequency and the same phase. So that is your so-called normal mode, and is a special kind of motion, which I would like you to identify with me. And we would later realize that those special kinds of motions, which are easier to understand, actually helps us to understand the general motion of the system.

You will realize that the most general motion of the system is just a superposition of all the identified normal modes. And then we are done, because we have a general solution already. So that's very good news. That tells us that we can understand the system systematically, and step by step. And then we can write the general motion of the system as a superposition of all the normal modes.

So let's get started. So can you guess what are the possible normal modes of this system? So that means each part of the system is oscillating at the same frequency and the same phase. Can anybody and any one of you guess what can happen, each part of this is an oscillating at the same frequency? Yeah?

AUDIENCE: If the two masses on that side are displaced the same amount and then they're --

YEN-JIE LEE: Very good. So he was saying that now I displace the right hand side two masses all together by a fixed amount, and also the left hand side, right, by a fixed amount and then let go. So that's what you're saying, right? OK. So the first mode we have identified is like this. So you have left hand side mass displace by delta x. And the right hand side two masses are also displaced by delta x. So basically you hold this three little masses and stretch it by the same-- introduce the same amplitude to all those three little masses, and let go. So that is actually one possible mode.

And if we do this, then basically what you are going to see is that this is actually roughly equal to this system. They're connected to each other by two springs. And the right hand side part of the system, both masses are oscillating at the same amplitude and the same phase. They look like as if they are just single mass with mass equal to 2m. And if you introduce a displacement of delta x, then what is going to happen is that if I take a look at the mass, left hand side mass, and the force acting on this mass, the force will be equal to minus 2k times 2 delta x, because that's the amount of stretch you introduce to the spring. And that will give you minus 4k delta x.

And we have already solved this kind of problem in the first lecture. So therefore you can immediately identify omega, in this case, omega a squared will be equal to 4k divided by 2m. This is actually the effective spring constant, and this is actually the mass. So that is actually the frequency of mode A.

Can you identify a second kind of motion which does that? So in this case, what is going to happen is that the three masses will-- OK, one, two, and three. The three masses will oscillate as a function of time like this with angular frequency of square root of 4k over 2m. What is actually a second possible motion? Yes?

AUDIENCE: All masses being stretched [INAUDIBLE]

YEN-JIE LEE: Compressed.

AUDIENCE: Compressed the same--

YEN-JIE LEE: Very good. I'm very lucky that I'm in front of such a smart crowd today. And we have successfully identified the second mode, mode B. So what is going to happen is that the left hand side mass is not moving. And you compress the upper one slightly and you stretch the lower one, the lower little mass to the opposite direction. The displacement is delta x, and the displacement of the second mass is delta x.

So what is going to happen? What is going to happen is that the left hand side mass will not move at all because the force, the spring force, acting on this mass is going to cancel. And apparently, these two little masses are going to be doing harmonic motion. Since this left hand side mass is not moving, it's as if this is a wall and this were a single spring, k, that's connected to a little mass. And it got displaced by delta x.

So what will happen is that this mass will experience a spring force, which is F equal to minus k delta x. Therefore, we can immediately identify omega b squared will be equal to k over m. So you can see that we have identified two kinds of modes, which every part of the system is oscillating at the same frequency and the same phase.

Everybody agree? No not everybody agree. Look at this guy this guy is not moving. How could this be? This is not the normal mode. Isn't it? OK. I hope that will wake you up a bit. I can be very tricky here. I can say that this mass is also oscillating, but with what amplitude?

AUDIENCE: Zero.

YEN-JIE LEE: Zero amplitude. Right So the conclusion is that, aha, everybody is actually oscillating at the same frequency, but these guy with zero amplitude.

AUDIENCE: Are they oscillating at the same phase as well?

YEN-JIE LEE: Yeah. Oh very good question. Another objection I receive. So life is hard for me today. Hey. This guy is oscillating out of phase. These two guys are out of phase. But I can argue that the amplitude of the first mass is actually has a minus sign compared to the second mass. Then they are again in phase. So very good. I like those questions. And I hope I have convinced you that everybody is oscillating, although you cannot see it, because the amplitude is small, is zero. And they are all oscillating at the same phase. Yes.

AUDIENCE: How come there's only one mass?

YEN-JIE LEE: Oh, the right hand side?

AUDIENCE: Yeah.

YEN-JIE LEE: Oh, yeah. That is because the left hand side mass, the 2m one, is actually not moving. Because they are two spring forces, one is actually pushing the mass, the other one's pulling the mass. And they cancel perfectly. Therefore, it's as if those two guys are not-- they don't find each other. And then it's like, they are just tools mass connected to a wall along. And then you can now identify what is the frequency. OK. Very good.

So we make the made a lot of the progress from the discussion. And now I would like to ask you for help. What is the third oscillation? Yes?

AUDIENCE: There's no third normal mode.

YEN-JIE LEE: There's no third normal mode.

AUDIENCE: There's no third normal mode because there are restricted to one dimension. I can not imagine another mode that would not displace the central mass.

YEN-JIE LEE: Very good. That's very good. On the other hand, you can also say, I also take the center mass motion as one of the normal mode. I think that's also fair to do that. Very good observation. You can see that the whole can move simultaneously. I can also argue that they are oscillating at the same frequency and the same phase, because they are all moving together.

So these are the 2m connected to mass one. All of them are moving in the same direction. So now I can calculate the force. What is the force? F is 0. Therefore, omega c is 0. So you can the small limit of omega. So of course, I can pretend that those mass are connected to a really, really small spring to the wall with is a spring constant k'. And I have k' goes to zero. And they are actually going to oscillate with omega c goes to zero.

So in this case, the amplitude is going to increase forever, because you have A sin omega c t. And this roughly A omega c t. And this is just vt. So what I want to argue is that this is actually also oscillation, but with angular frequency zero. And the general motion can be in written as vt times c, for example, some constant. Any questions?

So what I'm going to do next may amaze you. Very good. So we have identified three different kinds of modes. We have mode A, which is with omega a squared equal to omega a squared. Where is omega a squared. There. It's 4k over 2m. And also, the motion is like this. x1 equal t A cosine omega a t plus phi a. x2 is equal to minus A, because they have different sine. So if the motion is in the left hand side direction, then the two masses are oscillating in the opposite direction. So therefore, I get a minus sign in front of A. Cosine omega a t plus phi a. x3 will be also equal to minus A cosine omega a t plus phi a.

Of course, I need to define what this x1, x2, x3. That's why most of you got super confused. So the x1, what I mean is that is that the displacement of the mass 2m, I call it x1. The displacement of the upper mass, the upper little mass, I call it x2. And finally, the displacement of the third mass, I call it x3. Therefore, you can see that mode A, you have this kind of motion. The amplitude of the first mass is A. Therefore, if I define that to be A, then the second and third one, or the amplitude will be defined as minus A. And you can see that all of them are oscillating at fixed angular frequency, omega a, omega a, omega a; and also fixed phase, phi a, phi a, phi a.

Of course, we can also write down what we get for mode B. For mode B, the left hand side mass is not moving, stay put. And the other two masses are oscillating at the frequency of omega b. And amplitude, they differ by a minus sign. OK. Omega b squared is equal to k over m from that logical argument. And then we get x1 equal to 0 times cosine omega b t plus phi b. x2, I get B cosine omega b t plus phi b. x3, I get minus B cosine omega b t plus phi b. Any questions here? Finally, mode C. All the mass, x1 is equal to x2 is equal to x3, is equal to C plus vt.

So you can see that we have identified three modes, mode A, mode B, and the mode C. And there are three angular frequencies which we identified for all of those normal modes, omega a, omega b, and omega c.

And you can see that we also identified how many free parameters. One free parameter, two, three, four, five, and six. If you careful, you write down the equation of motion of this system, you will have three coupled differential equations. And those are second order differential equations. If you have three variables, three second order differential equations.

If you manage it magically, with the help from a computer or from math department people, how many free parameter would you expect in you a general solution? Can anybody tell me well how many? I have three second order differential equations. Yes?

AUDIENCE: 6?

YEN-JIE LEE: 6. So look at what we have done we identified already 1, 2, 3, three normal modes. By there are 1, 2, 3, 4, 5 6, six free parameters. That tells me I am done. I'm done. Because what is the general solution? The general solution is just a superposition of mode A mode B and mode C. You have six free meters to be determined by six initial conditions, which I would like-- I have to tell you what are those initial conditions.

So isn't this amazing to you? I didn't even solve the differential equation, and I already get the solution. And you can see another lesson we learned from here is that, oh no, you can imagine that the motion of the system can be super complicated. This whole thing can do this, all the crazy things are all displaced, and the center of mass can move, as you said. But the result is actually very easy to understand. It's just three kinds of motion, the displacement, and two kinds of simple harmonic motion. We add them together. And then you get the general description of that system.

So everything is so nice. We understand the motion of that system. But in general, life is very hard. For example, now I do something crazy here. I change this to 3. What are the normal modes? Can anybody tell me? It becomes very, very difficult, because there's no general symmetry of that kind of system. So we are in trouble. One of the modes maybe still there, which is actually mode B. But the other modes are harder to actually guess. So you can see that that already brings you a lot of trouble.

And you can see that I can now couple not just two objects, I can couple three objects, four objects, five objects, 10 objects. Maybe I will put that in your p set and see what happens. And you can see that this becomes very difficult to manage. So what I'm going to do in the rest of this lecture is to introduce you a method which you can follow in general to solve the question and get the normal mode frequencies and normal modes.

So we will take a four minute break. And we come back at 12:20. So if you have any questions, let me know.

What we are going to do in the following exercise is to try to understand a general strategy to solve the normal mode frequencies and the normal mode amplitudes, so that you can apply this technique to all kinds of different systems.

So what I am going to do today is to take these three mass system, and of course as usual, I try to define what is this coordinate system? The coordinate system I'm going to use is x1 and x2 an x3 describing the displacement of the mass from the equilibrium position. And the equilibrium means that there's no stretch on the spring. The string is unstretched. It's at their own natural length, l0. So once I define that, I can do a force diagram analysis.

So that starts from the left hand side mass with mass equal to 2m. I can write down the equation of motion, 2m x1 double dot. And this is going to be equal to k x2 minus x1 plus k x3 minus x1. So there are two spring forces acting on this mass, the left hand side mass. The first one is the upper spring. The second one is coming from the lower one.

And you can see that both of them are proportional to spring constant k, and also proportional to the relative displacement. And you can see that the two relative displacement, which is the amount of stretch to the spring, is actually x2 minus x1, and the x3 minus x1. Am I going too fast? OK. Everybody's following.

And you can actually check the sign. So you may not be sure. Maybe this is your x1 minus x2. But you can check that, because if you increase x1, what is going to happen? This term will become more negative. More negative in this coordinate system is pointing to the left hand side. So that makes sense. Because if I move this mass to the right side, then I am compressing the springs. Therefore, they are pushing it back. Therefore, this is actually the correct sign, x2 minus x1. The same thing also applies to the second spring force. So that's a way I double check if I make a mistake.

Now, this is actually the first equations of motion. And I can now also work on a second mass. Now I focus on a mass number two. The displacement is x2. Therefore the left hand side of Newton's Law is m x2 double dot. And that is equal to the spring force. The spring force, there's only one spring force acting on the mass. Therefore, what I am going to get is k x1 minus x2. Everybody's following?

You can actually check the sign carefully, also. And finally, I have the third mass, very similar to mass number two. I can write down the equation of motion, which should k x1 minus x3. So that is my coupled second order differential equations.

There are three equations. And all of them are second order equations. So this looks a bit messy. So what I'm going to do Is no magic. I'm just collecting all the terms belonging to x1, and put them together, all the terms belonging to x2, putting all together, and just rearranging things. So no magic. So I copied this thing, left hand side. 2m x1 double dot. And the dot will be equal to minus 2k x1 plus, I collect all the times related to x2, plus k x2, there's only one term here, then plus k x3. I'm just trying to organize my question.

So you can see that I collect all the terms related to x1 to here. Minus k, minus k, I get minus 2k. And the plus k for x2, plus k for the x3. And I can also do that for m x2 double dot. That will be equal to k x1 minus k x2 plus zero x3, just for completeness.

I can also do the same thing for the third mass, m x3 double dot. This is equal to k x1 plus 0 x2. There's no dependence on x2, because x1 and x2-- x3 and x2 are not talking to each either directly. Finally, I have the third, which is minus k x3.

Now our job is to solve those coupled equations. Of course, you have the freedom, if you know how to solve it yourself, you can already go ahead and solve it. But what I am going to do here is to introduce technique, which can be useful for you and make it easier to follow. It's a fixed procedure.

So what I can do is the following. I can write everything in them form of a matrix. How many of you heard the matrix for the first time? 1, 2, 3, 4. OK. Only four. But if you are not being familiar with matrix, let me know, and I can help you. Let the TA know. And also, there's a section in the textbooks, which I posted on announcement, which is actually very helpful to understand matrices. But sorry to these four students, we are going to use that. And maybe, you already learn how it works from here.

So one trick which we will use in this class is to convert everything into matrix format. What I am going to do is to write everything in terms of M, capital X, capital M, capital X double dot equal to minus capital K capital X. Capital M, capital X, and capital K, those are all matrices. Because I write this thing I really carefully, therefore we can already immediately identify what would be M, capital M and capital X and a K.

So I can write down immediately will be equal to 2m, 0, 0, 0, m, 0, 0, 0, m. Because there's only one in each line, you only have one term. X1 double dot, x2 double dot, x3 double dot. And also, you can write down what will be the X. This is actually a vector. X will be equal to x1, x2, x3.

Finally, you have the K? How do I read off K? Careful, there's a minus sign here, because I would like to make this matrix equation as if it's describing a simple harmonic motion of a one dimensional system. So it looks the same, but they are different because those are matrices. But therefore, I have in my convention I have this minus sign there. Therefore, when you read off the K, you have to be careful. So what is K? K is equal to 2k. You have the minus 2k here in front of x1. But because I have a minus sign there, therefore this one is actually taken out. So we have 2k there. Then you have minus k, minus k, minus k, k, 0. Minus k, plus k, 0. And finally, you can also finish the last row. You get minus k, 0, k. K becomes minus k. Minus k becomes k.

So we have read off all those matrices successfully. So you may ask, what do they mean? Do they get the meaning? M, K, X, what those? M, capital M matrix, tells you the mass distribution inside the system. So that's the meaning of this matrix. X is actually vector, which tells the position of individual components in the system. Finally, what is K? K is telling you how each component in the system talks to the other components. So K is telling you the communication inside the system.

So now we understand a bit what is going on. And as usual, I will go to the complex notation. So I have xj, the small xj are the position of the mass, x1, x2, and x3. xj will be real part of small zj. Small xj equal to real part of zj. Therefore, I can now write everything in terms of matrices again. So now I can write the solution to be Z, the capitol z is a matrix, exponential i omega t plus phi. This is the guess the solution I have. A1, A2, and A3. Those are the amplitudes, amplitude A of the first mass, amplitude of the second mass, amplitude of the third mass, in their normal mode.

And all of those are oscillating at the same frequency, omega, and the same phase, phi. Does that tell you something which we learned before? Oh, that's the definition of the normal mode. I'm using the definition of the normal mode. Every part of the system is oscillating at the same frequency in the same phase. And we use that to construct my solution. The complex version is exponential i omega t plus phi, oscillating at the same frequency, oscillating at the same phase. And those are the amplitude, which I will solve later. OK, any questions? I hope I'm not going too fast.

If everybody can follow, now I can go ahead and solve the equation in the matrix format. So now I go to the complex notation. So the equation M X double dot equal to minus KX becomes M Z double dot equal to minus KZ. And also, I can immediately get the Z double dot will be equal to minus omega squared Z, because each time I do a differentiation, I get i omega out of the exponential function. And I cannot kill that exponential function, so it's still there. Therefore, I get minus omega squared in front of Z. I hope that doesn't surprise you.

So that's very nice and very good news. That means I can replace this Z double prime with minus omega squared Z. Then I get minus M omega squared Z. And this is equal to minus KZ. And I can cancel the minus sign. That becomes something like this.

So, I can now cancel the exponential i omega t plus phi, because I have Z in the left hand side. And exponential i omega t plus phi is just a number. So therefore, I can cancel it. So what is going to happen if I do that? Basically, what I'm going get is I get M omega squared A equal to K A. I'm trying to go extremely slowly, because this is the first time we go through matrices.

So now you have this expression. Left hand is a matrix, M, times some constant, omega squared. I can actually get omega squared in front of it, because this is actually just a number. A is just a vector, which is A1, A2, A3, also a matrix. K is actually how the individual components talks to the others. So that's there, times A.

Now I would like to move everything to the right hand side, all the matrices in front A to the right hand side. Then I multiply both sides by M minus 1. So I multiply M minus 1 to the whole equation. M minus 1, what is M minus 1? The definition is that the inverse of M is called M minus 1. M minus 1 times M is equal to I, which is actually 1, 1, 1.

Therefore, if I do this thing, then I would get omega squared. M minus 1 times M becomes I, unit matrix. And this is equal to M minus 1 K A. And be careful, I multiply M minus 1, the inverse of M, from the left hand side. That matters. So now I can move everything to the same side. I moved the left hand side term to the right hand side. Therefore, I get M minus 1 K minus omega squared I. Those are all matrices. Times A, this is equal to 0. Any questions?

So a lot of manipulation. But if you think about it, and you are following me, you'll see that all those steps are exactly identical to what we have been doing for a single harmonic oscillator. Looks pretty familiar to you. But the difference is that now we are dealing with matrices.

AUDIENCE: What is A?

YEN-JIE LEE: Oh, A. A is actually this guy. I define this to be A. And that means Z will be exponential i omega t plus phi times A. I didn't actually write it explicitly. But that's what I mean. Any more questions? Yes?

AUDIENCE: [INAUDIBLE] is for [INAUDIBLE]?

YEN-JIE LEE: Can you repeat that?

AUDIENCE: So this whole process, this is mode A, right?

YEN-JIE LEE: Yeah. So this whole process is for, not really the for mode A. So that A may be confusing. But in general, if I have a solution, and I assume that the amplitude can be described by a matrix. So it's in general. And you'll see that we can actually derive the angular frequency of mode A, mode B, and mode C afterward. I hope that answers your question. So you see that for in general, what I have been doing is that now, all those things are equivalent to the original equation of motion. What I am doing is purely cosmetic. You see, make it beautiful. So all those things, this thing is exactly the equivalent to that thing, up there. Up to M X double dot equal to minus K x. Cosmetics. Beautiful. Looks-- I like it. All right.

Then what I have been doing is that now I introduce using a definition of normal mode. I guess the solution will have this functional form. Z equals to exponential i omega t plus phi, everybody oscillating at the same frequency, the same phase. Frequency omega, phase phi. And everybody can have different amplitude. You can see from this example, normal modes, they can have different amplitude. The amplitude is what? I don't know yet. But we will figure it out.

Then that's my assumption. The definition of normal mode. And I plug in to the equation of motion. Then this is what we are doing to simplify the equation of motion. There's no magic here. If I plug in the definition on normal mode to that equation, this is actually going to bring you to this equation, matrix equation.

So if you have learned matrices before, you have something, some matrix, times Z. This is equal to zero. A is not zero. I hope. If it's zero, then the whole system is not moving. Then it's not fun. So if A is not zero, then this thing should be-- this thing times A should make this equation equal to 0. So what is actually the required condition? I get stuck,

and of course again, my friend from math department comes to save me.

That means if this thing has a solution, this equation has a solution, that means that determinant of M minus 1 K minus omega squared I has to be equal to 0. So that is the condition for this equation to satisfy this to be equal to 0. And just to make sure that I don't know what is the angular frequency omega yet. I don't know what is the phi yet. We can actually solve the angular frequency, omega.

So now, turn everything around. And basically now, using this normal mode definition, and some mathematical manipulation, the condition we need for this equation to satisfy equal to 0, is determinant M minus 1 K minus omega squared I.

I can write down M minus 1 K minus omega squared I explicitly, just to help you with mathematics. M minus 1 K is equal to 1 over 2m, 0, 0, 0, 1/m, 0, 0, 0, 1/m. It's just the inverse matrix of the M matrix. Therefore, now I can write down the explicit expression of M minus 1 K minus omega squared I. This will be equal to k over m minus omega squared, minus k over 2m, minus k over 2, minus k over m, k over m minus omega squared, 0. I will write down all the elements first. Then I will explain to you how I arrived at the expression.

OK. So this is M minus 1 k. The definition of M minus 1 is that. And the definition of K is in the upper right corner of the black board. Therefore, if you multiply M minus 1 K, basically, the first column will get-- wait a second. Did I make a mistake? No. OK. So basically, what you arrive at is k/m, k/m, k/m. And also, the minus k over 2m for the rest part of the matrix. And the minus omega squared I will give you the diagonal component. Yes?

AUDIENCE: Why do you have to take the determinant and set it equal to 0 instead of just setting that equal to zero?

AUDIENCE: This is a matrix. So these are the matrix. So a matrix times A will be equal to zero. The general condition for that to be satisfied is more general. It's actually the determinant of this matrix equal to zero. Because this is actually multiplied by some back to A.

So I think there are mathematical manipulation. Basically, you would just collect the terms. And then calculate M minus 1 K first. And the minus omega squared I will give you all the diagonal and terms have a minus omega square there. And that is actually the matrix. And of course, I can calculate the determinant. So if I calculate the determinant, then basically I get this times that times that. So what you get is k over m minus omega squared times k over m minus omega squared times k over m minus omega squared. So these are all diagonal terms. And the minus 1 over 2 k squared over m squared, k squared over m squared. sorry. Minus omega squared.

So that's this off diagonal term, this times this times that. OK. It will give you the second term. And the third one, which survived because of those zeros, many, many terms are equal to 0. And then the third term, which is nonzero, is again minus 1 over 2k squared over m squared, k over m minus omega squared. And this is actually equal to 0, because the determinant of this matrix is equal to zero. Everybody following? A little bit of a mess. Because I have been doing something very challenging. I'm solving a 3 by 3 matrix problem in front of you right. So the math can get a bit complicated. But next time, I think we are going to go to a second order one, 2 by 2 matrix. And I think that will be slightly easier.

But the general approach is the same. So basically, you calculate M minus 1 K minus omega squared. Then you get what is inside, all the content inside this matrix. Then you would calculate the determinant. And basically, you can solve this equation. Now I can define omega0 squared to be k/m. And I can actually make this expression much simpler.

Then basically, what you are getting is omega0 squared minus omega squared to the third minus 1/2 omega0 to the fourth, omega0 squared minus omega squared. Minus 1/2 omega0 to the fourth, omega0 to the square, minus omega squared. And this is equal to 0. And you can factor out the common components. Then basically, what you are going to get is, you can write this thing to be omega0 squared minus omega squared, omega squared. Because all of them have omega squared. And omega squared minus 2 omega0 squared. And that's equal to 0.

So I am skipping a lot of steps from this one to that one. But in general, you can solve this third order equation. And I can first combine all those terms together. And then I factor out the common components. Then basically, what you are going to arrive at is something like this. A lot of math here. But we are close to the end.

So you can see now what are the possible solutions for omega. That is the omega, unknown angular frequency we are trying to figure out. You can see that there are three possible omegas that can make this equation equal to 0. The first one is omega equal to omega0. The second one is square root of omega 0, coming from this expression, that omega squared minus 2 omega zero squared. If omega equal to square root 2 omega0, this will be equal to zero. And that will give you the whole expression equal to 0. Then finally, I take this term. And then you will get zero. Omega squared, if omega is equal to 0, then the whole expression is 0.

I have defined omega0 squared to be equal to k over m. Therefore, I can conclude that omega squared is equal to k over m, 2k over m, and 0. Look at what we have done, a lot of mathematics. But in the end, after you solve the eigenvalue problem, or the determinant equal to zero problem, you arrive at that there are only three possible values of omega which can make the determinant of M minus 1 K minus omega squared I equal to 0.

What are the three? k/m, 2k/m, and 0. If you look at this value, then we'll say, this is essentially what we actually argued before, right? Omega A squared is equal to 4k over 2m is 2k over m. Wow. We got it.

The second one is, we think about really keep a straight question in my head and understand this system. The second identified normal more is having omega squared be equal to k/m. I got this also here magically, after all those magics. And finally, the third one, the math also knows physics. It also predicted that this is one mode which have oscillation frequency of 0. Isn't that amazing to you?

But that also gives us a sense of safety. Because I can now add 10 pendulums, or 10 coupled system to your homework, and you will be able to solve it.

So very good. This example seems to be complicated. But the what I want to say, I have one minute left, is that what we have been doing is to write the equation of motion based on force diagram. Then I convert that to matrix format, and X double dot equal to minus KX. Then I follow the whole procedure, solve the eigenvalue problem. Then I will be able to figure out what are the possible omega values which can satisfy this eigenvalue problem or this determinant. M minus 1 K minus omega squared I equal to 0 problem. And after solving all those, you will be able to solve the corresponding so-called normal mode frequencies. You can solve it. And of course, you can plug those normal mode frequencies back in, then you will be able to derive the relative amplitude, A1, A2, and A3.

So what we have we learned today? We have learned how to predict the motion of coupled oscillators. That's really cool. And then next time, we are going to learn a special kind of motion in coupled oscillators, which is the big phenomena. And also, what will happen if I start to drive the coupled oscillators? So I will be here if you have any questions about the lecture. Thank you very much.