Symmetry : A very important concept in Physics, Mathematics and art!

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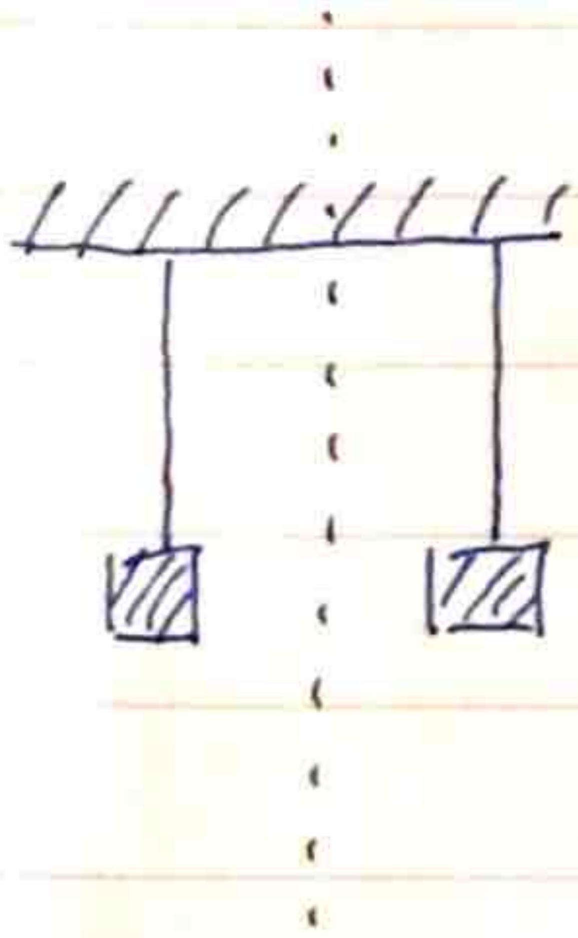
In the example we just discussed: we solved the eigenvalue problem $(M^{-1}K)A = \omega^2 A$

We get normal mode frequencies, and amplitude ratios.

In fact we can find the normal modes much easier by symmetry!



This system is "invariant" under reflection! (Physics is unchanged)

$x_1 \to -x_2$

$x_2 \to -x_1$

This means that

if $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is a solution

$\tilde{X}(t) = \begin{pmatrix} -x_2 \\ -x_1 \end{pmatrix}$ is also a solution

To describe it mathematically we define $S \equiv \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Symmetry matrix

$$\tilde{X}(t) = S\,X(t)$$

Original Equation of Motion:

$$\ddot{x}(t) = -M^{-1}K\,x(t) \quad ①$$

$x(t)$ is a solution $= A\cos(\omega t+\phi)$

If $\tilde{x}(t) = S\,x(t)$ ② is also a solution

$$\Rightarrow \ddot{\tilde{x}}(t) = -M^{-1}K\,\tilde{x}(t)$$

$$② \Rightarrow S\ddot{x}(t) = -M^{-1}KS\,x(t)$$

$$\updownarrow$$

We also know

$$S\times(①) \quad S\ddot{x}(t) = -SM^{-1}K\,x(t)$$

Therefore, if $SM^{-1}K = M^{-1}KS \Rightarrow \tilde{x}$ is also a solution

or usually we say "commute" $[A,B] \equiv AB - BA = 0$

$$[S, M^{-1}K] = 0$$

If $x(t) = A^{(1)}\cos(\omega_1 t)$ and $\omega_1 \neq \omega_2$

$$\Rightarrow \tilde{x}(t) \propto A^{(1)}\cos(\omega_1 t)$$

$\because$ any solution oscillating with ω_1 must be proportional to $A^{(1)}$

$$\Rightarrow S\,x(t) = SA^{(1)}\cos(\omega_1 t) \propto A^{(1)}\cos\omega_1 t$$

$$\Rightarrow SA^{(1)} = \beta_1 A^{(1)}$$

similarly $SA^{(2)} = \beta_2 A^{(2)}$

$A^{(1)}$ and $A^{(2)}$ are also eigenvectors of S !!!

(Solutions of eigenvalue problem $SA = \beta A$) !!

Now we can run the logic in the opposite direction: Given (i) $SA=\beta A$

(ii) $[S, M^{-1}K]=0$

$$\hookrightarrow \; S M^{-1}KA \underset{(ii)}{=} M^{-1}KSA \underset{(i)}{=} \beta M^{-1}KA$$

$\Rightarrow$ $M^{-1}KA$ is an eigenvector of S with the same eigenvalue as A

If the eigenvalues of S are all different

$$\Rightarrow \; M^{-1}KA \propto A$$

$\Rightarrow$ A is also an eigenvector of $M^{-1}K$!!!

Now instead of solving

$$M^{-1}KA^{(n)} = \omega_{(n)}^2 A^{(n)}$$

We can solve

$$SA^{(n)} = \beta_n A^{(n)}$$

which is much easier!

We know $S^2 = I$

$$S^2 A^{(n)} = \beta_n^2 A^{(n)} = A^{(n)}$$

$$\Rightarrow \; \beta_n = \pm 1 \quad !!!$$

Use β_n $\Rightarrow$ Get $A^{(n)}$ $\Rightarrow$ $\beta_1 = -1 \Rightarrow A^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\beta_2 = 1 \Rightarrow A^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow$ Use $M^{-1}KA^{(n)} = \omega_{(n)}^2 A^{(n)}$

We can again find $\omega_{(n)}^2$

What we have learned here:

(1) We can find the normal modes by solving the eigenvalue problem for the symmetry matrix S, instead of $M^{-1}K$

(Given $SM^{-1}K = M^{-1}KS$ or $[S, M^{-1}K] = 0$)

commute!

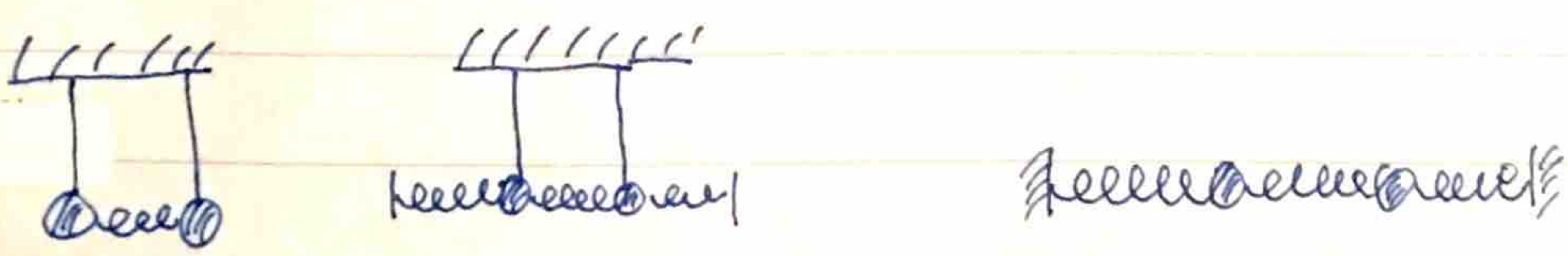
(2) This is a remarkable result :

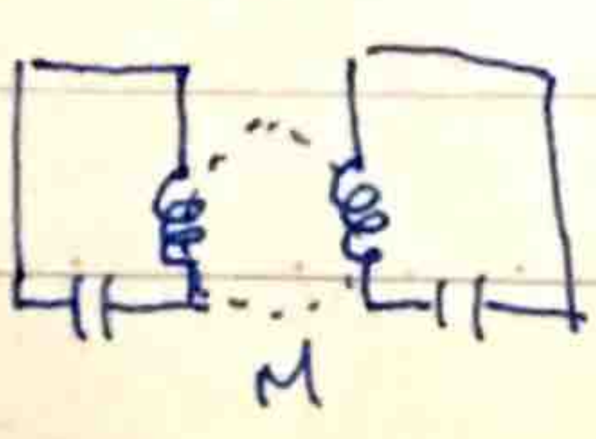
ALL SYSTEMS SATISFY (two-component system)

$$SM^{-1}K = M^{-1}KS$$

$\Rightarrow$ Have the same eigenvectors !!!!

Once you solved one, you solved all of them!





. . . . . All solved!

Learned : Symmetry and symmetry matrix S

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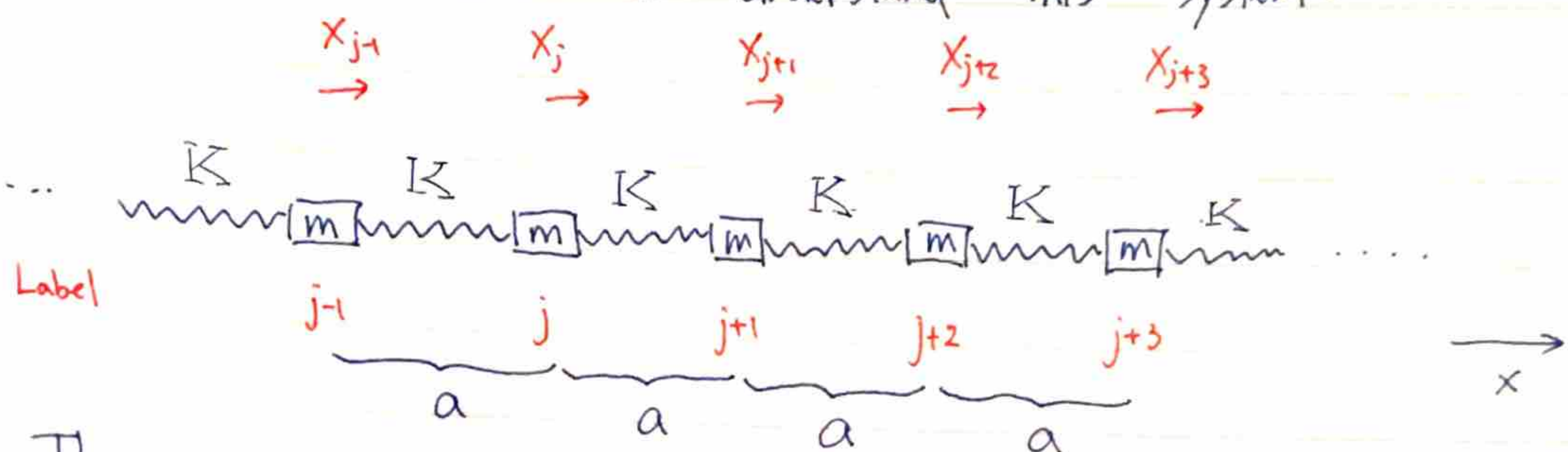
$\Longrightarrow$ infinite number of coupled oscillators

(DEMO)

Now we would like to understand this system :



The masses are constrained to move only in the x direction

Ideal springs : spring constant K and natural length a.

Equation of Motion: if we focus on the jth object

$$m\ddot{x}_j = -K(x_j - x_{j-1}) - K(x_j - x_{j+1})$$

$$= +Kx_{j-1} - 2Kx_j + Kx_{j+1}$$

$$x_j = A_j \cos(\omega t + \phi) = Re\left(A_j e^{i(\omega t+\phi)}\right)$$

$$\Rightarrow M = \begin{pmatrix} \ddots & & & & \\ m & 0 & 0 & 0 & \cdots \\ 0 & m & 0 & 0 & \cdots \\ 0 & 0 & m & 0 & \cdots \\ 0 & 0 & 0 & m & \cdots \\ \vdots & \vdots & \vdots & & m \end{pmatrix} \qquad M^{-1}K = \begin{pmatrix} \cdots & & & & & & & \\ \cdots 0 & \frac{-k}{m} & \frac{2k}{m} & \frac{-k}{m} & 0 & \cdots & & \\ \cdots 0 & 0 & \frac{-k}{m} & \frac{2k}{m} & \frac{-k}{m} & 0 & \cdots & \\ \cdots 0 & 0 & 0 & \frac{-k}{m} & \frac{2k}{m} & \frac{-k}{m} & \cdots & \\ & & & & & & & \ddots \end{pmatrix}$$

$$A = \begin{pmatrix} \vdots \\ A_j \\ A_{j+1} \\ A_{j+2} \end{pmatrix}$$

Problem: We don't know how to solve $\infty \times \infty$ matrix ....

use "Symmetry" !!

Symmetry is widely used in theoretical physics

A powerful tool : Simplify the problem

We discussed Reflection symmetry.

Now : Space translation symmetry

If we move the system by "a" to the left
$\Rightarrow$ the Physics is unchanged

$$A' = SA$$

where S matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ & & & & 1 & \\ & & & & & 1 \\ \vdots & & & & & & \ddots \end{pmatrix}$$

An infinite matrix with 1s along the next-to-diagonal.

We want to find the eigenvalues and eigenvectors of S

$$A' = SA = \beta A \qquad \beta: \text{eigenvalue.}$$

$$A = \begin{pmatrix} \vdots \\ A_j \\ A_{j+1} \\ A_{j+2} \\ \vdots \end{pmatrix} \qquad SA = \begin{pmatrix} \vdots \\ A_{j+1} \\ A_{j+2} \\ A_{j+3} \\ \vdots \end{pmatrix}$$

In this case:

$$A_j' = \beta A_j = A_{j+1}$$

We don't know yet what is β, but we know

if $A_0 = 1$

$A_1 = \beta$

$A_2 = \beta^2$ .... $A_j = \beta^j$

This works for all nonzero values of β

$\Rightarrow$ ∞ number of normal modes (and eigenvalues)

Make sense? Yes. Because we have infinite # of degrees of freedom!

Now we have the normal modes and eigenvectors.

From previous discussion:

If $[S, M^{-1}K] = 0$ and eigenvalues are all different

$\Rightarrow$ S and $M^{-1}K$ share the same eigenvectors.

To get the corresponding angular frequency:

$$M^{-1}KA = \omega^2 A$$

Use $M^{-1}K$ matrix $\Rightarrow$

$$\omega^2 A_j = -\frac{K}{m}A_{j-1} + \frac{2K}{m}A_j - \frac{K}{m}A_{j+1}$$

$$= \omega_0^2 \left(-A_{j-1} + 2A_j - A_{j+1}\right)$$

$$\boxed{\omega_0^2 \equiv \frac{K}{m}}$$

Since $A_j \propto \beta^j$

$$\Rightarrow \quad \omega^2 \beta^j = \omega_0^2 \left(-\beta^{j-1} + 2\beta^j - \beta^{j+1}\right)$$

$$\Rightarrow \quad \omega^2 = \omega_0^2 \left(\frac{-1}{\beta} + 2 - \beta\right)$$

$$\Rightarrow \omega^2 = \omega_0^2 \left(2 - (\beta + \beta^{-1}) \right) \quad \text{——(1)}$$

β can be any value.

Also, $\beta = b$ & $\beta = \frac{1}{b}$ gives the same angular frequency ω!

However if $|\beta| \neq 1 \Rightarrow$ amplitude goes to infinity since $A_j \propto \beta^j$.

$\Rightarrow$ Consider $|\beta| = 1$ case.

$\Rightarrow \beta = e^{ika}$ $\qquad A_j \propto e^{ijka}$

If we plug that in eq. (1)

$$\Rightarrow \omega^2 = \omega_0^2 \left(2 - (e^{ika} + e^{-ika}) \right)$$

$$\boxed{\omega^2 = 2\omega_0^2 (1 - \cos ka)}$$

"dispersion relation"

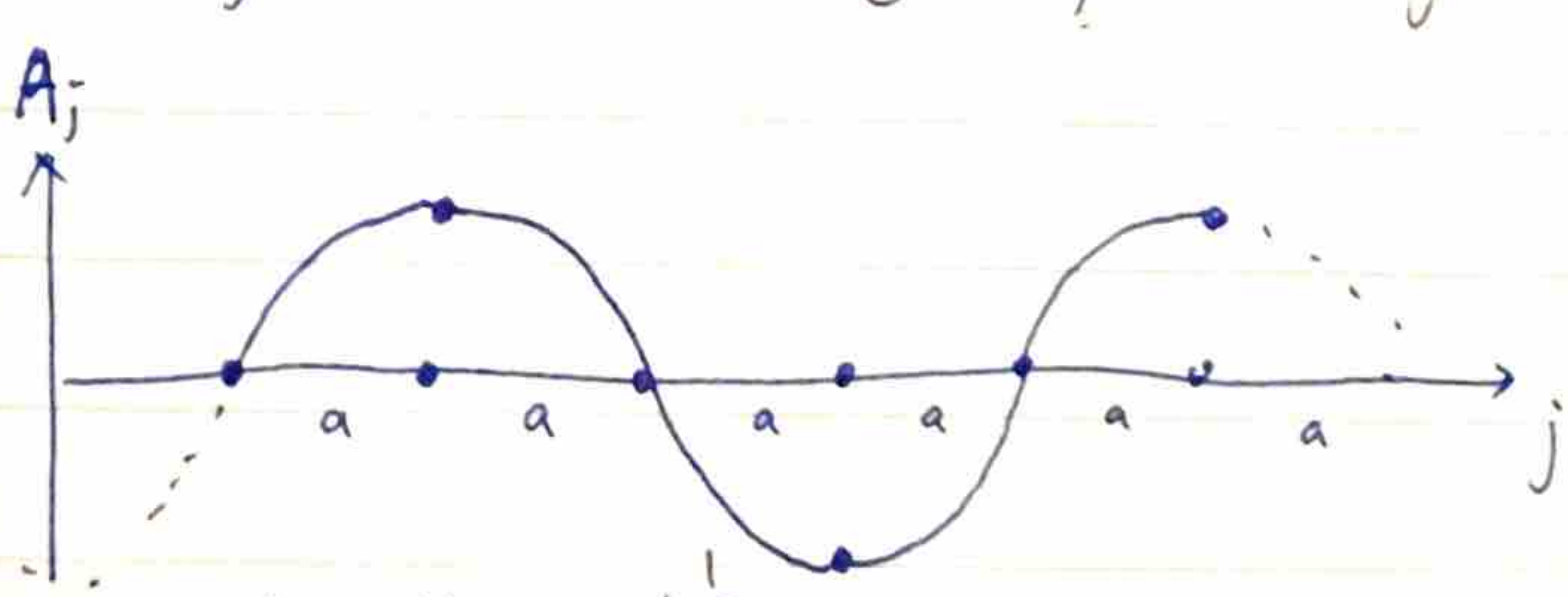
Let's take a look at this equation:

(1) There are infinite number of normal modes:

Each k gives a normal mode

(2) Amplitude: $A_j = \frac{1}{2i}\left(e^{ijka} - e^{-ijka}\right) = \sin jka$ To make A_j real

Ex: $k = \frac{\pi}{2a}$

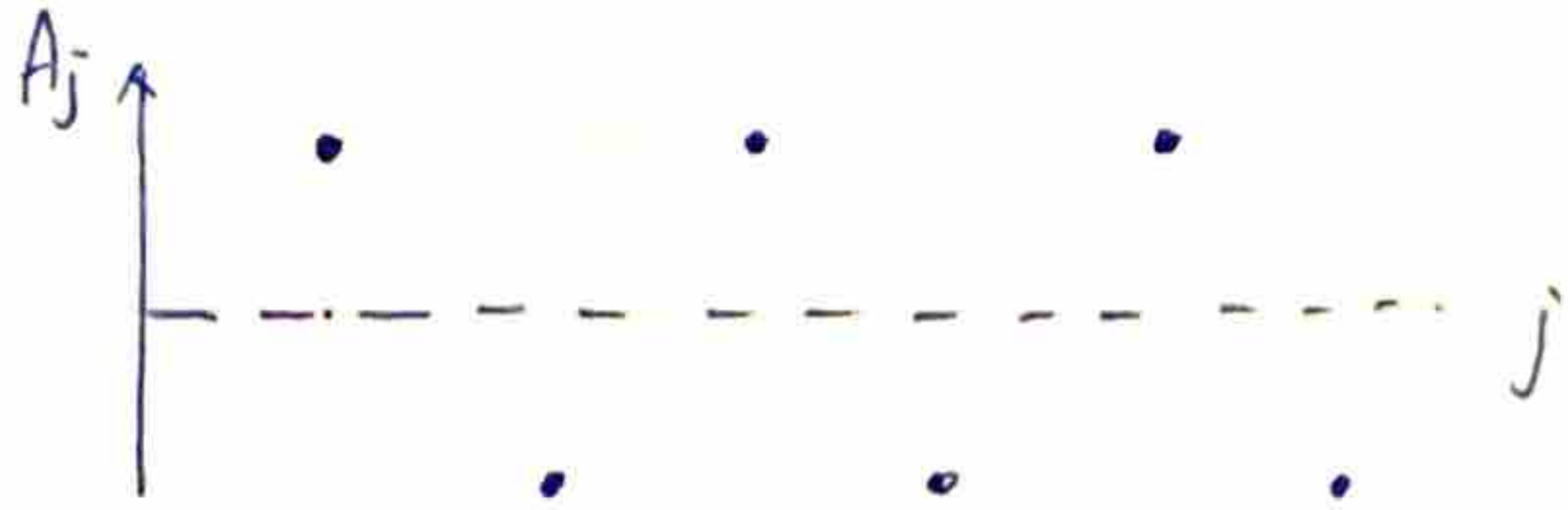


Since both $A^{\beta} = \beta^j = e^{ijka}$ and $A^{(\frac{1}{\beta})}$ are eigenvector of $M^{-1}K$ with eigenvalue $\omega^2 = 2\omega_0^2 (1 - \cos ka)$ $\Rightarrow$ linear combinations of them are also eigenvector of $M^{-1}K$!

See Georgi P. 113

(3) Maxima frequency: $\omega^2 = 4\omega_0^2$

$\cos(ka) = -1$



(4) $\cos(ka) = 1$ or $\beta = 1$

Minimum frequency: $\omega^2 = 0$ (all masses are moving in the same direction)!

All possible motions: linear combination of all normal modes

each normal mode: Standing waves.