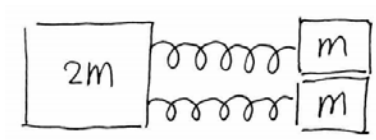
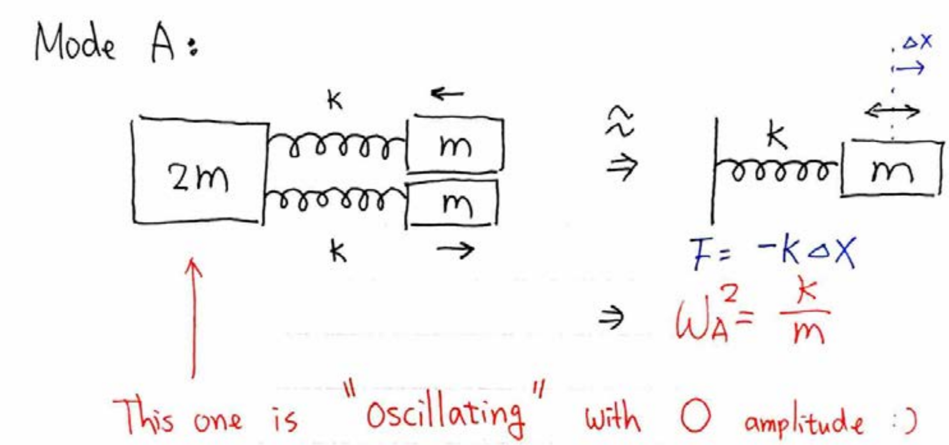
8.03 Lecture 4

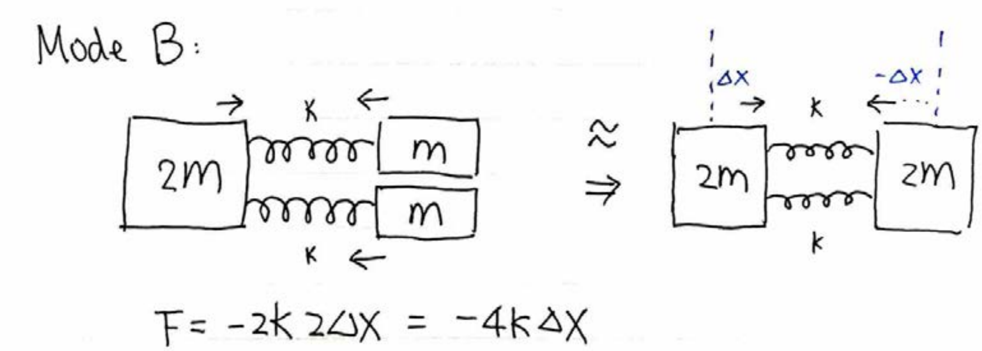
Coupled oscillators

In general, the motion of coupled systems can be extremely complicated. Editors note: watch the video lectures to see examples of complicated coupled oscillators.
Let's consider an example:

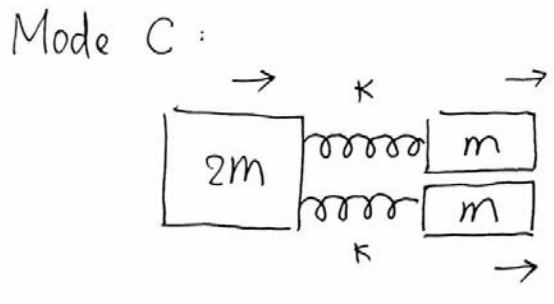


There are many kinds of motion in this system! If you stare at it long enough you an identify a special kind of motion! The **"normal mode:"** every part of the system is oscillating at the same phase and the same frequency. We will later realize the most general motion is a superposition of the normal modes. We can understand the system systematically step-by-step.
In general, coupled oscillators are complicated but there are easier cases we can solve, being guided by our physical intuition.
Can you guess the normal modes of this example?





Where in Mode A we have $\omega_A^2 = \frac{k}{m}$ and in Mode B we have $\omega_B^2 = \frac{4k}{2m} = \frac{2k}{m}$
Is there a Mode C? Yes!



Where there is no force, and $\omega_C = 0$ because there is no oscillation. The whole system is simply translating.

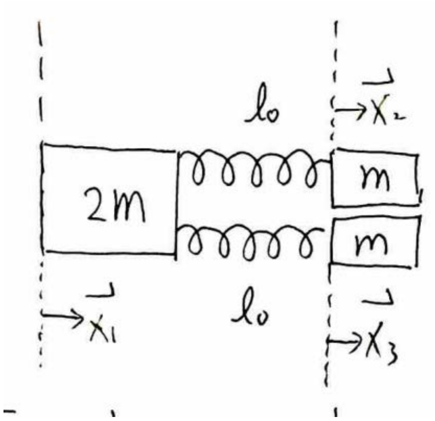
To summarize:

Mode A:
$$x_1 = 0$$
$$x_2 = A\cos(\omega_A t + \phi_A)$$
$$x_3 = -A\cos(\omega_A t + \phi_A)$$
Mode B:
$$x_1 = B\cos(\omega_B t + \phi_B)$$
$$x_2 = -B\cos(\omega_B t + \phi_B)$$
$$x_3 = -B\cos(\omega_B t + \phi_B)$$
Mode C:
$$x_1 = x_2 = x_3 = c + vt$$

Therefore the general solution is:

$$\begin{aligned} x_1 &= 0 + B\cos(\omega_B t + \phi_B) + c + vt \\ x_2 &= A\cos(\omega_A t + \phi_A) - B\cos(\omega_B t + \phi_B) + c + vt \\ x_3 &= -A\cos(\omega_A t + \phi_A) - B\cos(\omega_B t + \phi_B) + c + vt \end{aligned}$$

Where A, B, C, ϕ_A, ϕ_B and v are constants to be determined by the initial conditions. Here we have 3 second order differential equations with 6 unknown constants.



From the analysis of the force diagram analysis we get:

$$2m\ddot{x}_1 = k(x_2 - x_1) + k(x_3 - x_1)$$
$$m\ddot{x}_2 = k(x_1 - x_2)$$
$$m\ddot{x}_3 = k(x_1 - x_3)$$

We can reorganize:

$$\begin{array}{rcrcrcr} 2m\ddot{x}_1 & = & -2kx_1 & + & kx_2 & + & kx_3 \\ m\ddot{x}_2 & = & kx_1 & - & kx_2 & + & 0x_3 \\ m\ddot{x}_3 & = & kx_1 & + & 0x_2 & - & kx_3 \end{array}$$

Now our job is to solve the equations. It is possible to solve this coupled set of differential equations directly, but we can use a **matrix** as a tool to help us. We convert everything to matrices. Our equation of motion is now

$$M\ddot{X} = -KX$$

where:

$$M = \begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 2k & -k & -k \\ -k & k & 0 \\ -k & 0 & k \end{bmatrix}$$

We go to complex notation where $X_j = \text{Re}[Z_j]$ and $Z \equiv e^{i(\omega t+\phi)}A$ and A is a column vector (A_1, A_2, A_3)

Solving the equation of motion:

$$\begin{aligned} M\ddot{Z} &= -KZ \\ M\omega^2 Z &= KZ \\ M\omega^2 A &= KA \\ \omega^2 A &= M^{-1}KA \\ \Rightarrow (M^{-1}K{-}\omega^2 I)A &= 0 \end{aligned}$$

Where I is the identity matrix. To have a soluation we need to solve

$$\det[M^{-1}K - \omega^2 I] = 0$$

$$(M^{-1}K - \omega^2 I) = \begin{bmatrix} \frac{k}{m} - \omega^2 & \frac{-k}{2m} & \frac{-k}{2m} \\ \frac{-k}{m} & \frac{k}{m} - \omega^2 & 0 \\ \frac{-k}{m} & 0 & \frac{k}{m} - \omega^2 \end{bmatrix}$$

Define $\omega_0^2 \equiv k/m$

$$\det \begin{pmatrix} \omega_0^2 - \omega^2 & \frac{-\omega_0^2}{2} & \frac{-\omega_0^2}{2} \\ -\omega_0^2 & \omega_0^2 - \omega^2 & 0 \\ -\omega_0^2 & 0 & \omega_0^2 - \omega^2 \end{pmatrix} = 0$$

$$\begin{aligned} (\omega_0^2 - \omega^2)^3 - \frac{1}{2}\omega_0^4(\omega_0^2 - \omega^2) - \frac{1}{2}\omega_0^4(\omega_0^2 - \omega^2) &= 0 \\ (\omega_0^2 - \omega^2)(\omega_0^4 - 2\omega_0^2\omega^2 + \omega^4 - \omega_0^4) &= 0 \\ (\omega_0^2 - \omega^2)\omega^2(\omega^2 - 2\omega_0^2) &= 0 \end{aligned}$$

$$\Rightarrow \quad \omega = \omega_0, \sqrt{2}\omega_0, 0$$

$$\omega = \sqrt{\frac{k}{m}}\ ,\ \sqrt{\frac{2k}{m}}\ ,\ 0$$

We get the same result! To get the relative amplitude of a normal made: Plut in the normal mode frequency you get in the equation $(M^{-1}K - \omega^2 I)A = 0$. For example: take Mode B where $\omega = \omega_B = \quad \overline{2k/m}$

$$\begin{array}{rcrcrcr} 0 & = & 2kA_1 & + & kA_2 & + & kA_3 \\ 0 & = & kA_1 & + & kA_2 & + & 0 \\ 0 & = & kA_1 & + & 0 & + & kA_3 \end{array} \qquad \Rightarrow \quad A_1 = -A_2 = -A_3$$

In Mode B we had

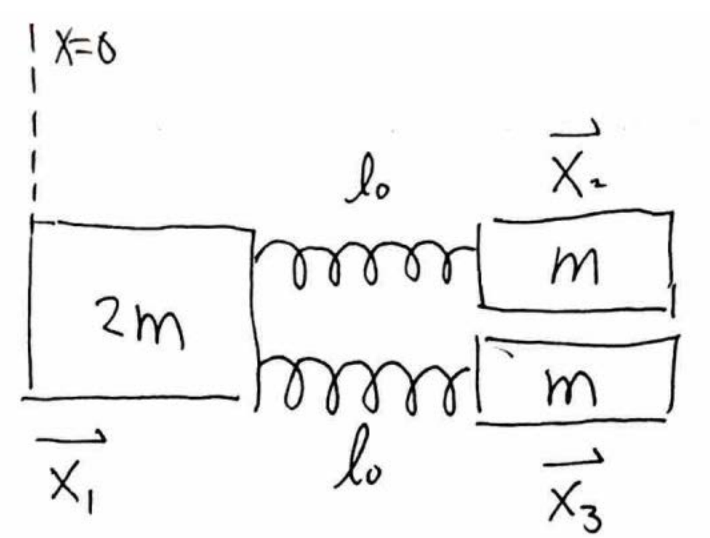
$$\begin{array}{rcl} x_1 & = & B\cos(\omega_B t + \phi_B) \\ x_2 & = & -B\cos(\omega_B t + \phi_B) \\ x_3 & = & -B\cos(\omega_B t + \phi_B) \end{array} \quad \text{or} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = B \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cos(\omega_B t + \phi_B)$$

It turns out this is just the simple harmonic oscillator!

Take Mode A as an example where $\omega = \omega_A = \sqrt{k/m}$. Plug in this frequency (into the equation $(M^{-1}K - \omega^2 I)A = 0$) to solve: $A_1 = 0$ and $A_2 = -A_3$

$$x_1 = 0$$
$$x_2 = A\cos(\omega_A t + \phi_A)$$
$$x_3 = -A\cos(\omega_A t + \phi_A)$$

There is an alternative way we can solve for the normal modes. We can define the length of the spring as l_0 and define a new origin:



From the analysis of the force diagram we get:

$$\begin{aligned} 2m\ddot{x}_1 &= k(x_2 - x_1 - l_0) + k(x_3 - x_1 - l_0) \\ m\ddot{x}_2 &= k(x_1 - x_2 + l_0) \\ m\ddot{x}_3 &= k(x_1 - x_3 + l_0) \end{aligned}$$

Redefine the x_2 and x_3 coordinates:

$$x_2' = x_2 - l_0 \qquad x_3' = x_3 - l_0$$

Now we have

$$\begin{aligned} 2m\ddot{x}_1 &= k(x_2' - x_1) + k(x_3' - x_1) \\ m\ddot{x}_2' &= k(x_1 - x_2') \\ m\ddot{x}_3' &= k(x_1 - x_3') \end{aligned}$$

Reorganizing:

$$\begin{array}{ccccccc} 2m\ddot{x}_1 & = & -2kx_1 & + & kx_2' & + & kx_3' \\ m\ddot{x}_2' & = & kx_1 & - & kx_2' & + & 0x_3' \\ m\ddot{x}_3' & = & kx_1 & + & 0x_2' & - & kx_3' \end{array}$$

Now use the definition of normal mode:

$$\begin{pmatrix} x_1 \\ x_2' \\ x_3' \end{pmatrix} = \text{Re}\left[\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} e^{i(\omega t + \phi)}\right]$$

$$\begin{aligned} -2m\omega^2 A_1 &= -2kA_1 + kA_2 + kA_3 & \qquad & 0 = (-2k + 2m\omega^2)A_1 + kA_2 + kA_3 \\ -m\omega^2 A_2 &= kA_1 - kA_2 + 0A_2 & \Rightarrow \quad & 0 = kA_1 + (m\omega^2 - k)A_2 + 0A_3 \\ -m\omega^2 A_3 &= kA_1 + 0A_2 - kA_3 & & 0 = KA_1 + 0A_2 + (m\omega^2 - k)A_3 \end{aligned}$$

Rewrite in matrix notation:

$$\begin{pmatrix} (2m\omega^2 - 2k) & k & k \\ k & (m\omega^2 - k) & 0 \\ k & 0 & (m\omega^2 - k) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

To get a solution we need to solve the equation where the determinant of the left matrix is zero.

$$\det \begin{pmatrix} (2m\omega^2 - 2k) & k & k \\ k & (m\omega^2 - k) & 0 \\ k & 0 & (m\omega^2 - k) \end{pmatrix} = 0$$

$$\begin{aligned} (2m\omega^2 - 2k)(m\omega^2 - k)^2 - 2k^2(m\omega^2 - k) &= 0 \\ (m\omega^2 - k)\left[(2m\omega^2 - 2k)(m\omega^2 - k) - 2k^2\right] &= 0 \\ (m\omega^2 - k)\omega^2(2m^2\omega^2 - 4km) &= 0 \end{aligned}$$

$$\omega = \sqrt{\frac{2k}{m}}\ ,\ \sqrt{\frac{k}{m}}\ ,\ 0$$

Get the same result!