

Formula Sheet Final Exam

Springs and masses:

$$m \frac{d^2}{dt^2} x(t) + b \frac{d}{dt} x(t) + kx(t) = F(t)$$

More general differential equation with harmonic driving force:

$$\frac{d^2}{dt^2} x(t) + \Gamma \frac{d}{dt} x(t) + \omega_0^2 x(t) = \frac{F_0}{m} \cos(\omega_d t)$$

Steady state solutions:

$$x_s(t) = A \cos(\omega_d t - \delta)$$

where

$$A = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \omega_d^2 \Gamma^2}}$$

and

$$\tan \delta = \frac{\Gamma \omega_d}{\omega_0^2 - \omega_d^2}$$

General solutions:

For $\Gamma = 0$ (undamped system):

$$x(t) = R \cos(\omega_0 t + \theta) + x_s(t)$$

where R and θ are unknown coefficients.

For $\Gamma < 2\omega_0$ (under damped system):

$$x(t) = R e^{-\frac{\Gamma}{2}t} \cos\left(\sqrt{\omega_0^2 - \frac{\Gamma^2}{4}} t + \theta\right) + x_s(t)$$

where R and θ are unknown coefficients.

For $\Gamma = 2\omega_0$ (critically damped system):

$$x(t) = (R_1 + R_2 t) e^{-\frac{\Gamma}{2}t} + x_s(t)$$

where R_1 and R_2 are unknown coefficients.

For $\Gamma > 2\omega_0$ (over damped system):

$$x(t) = R_1 e^{-\left(\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}\right)t} + R_2 e^{-\left(\frac{\Gamma}{2} - \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}\right)t} + x_s(t)$$

where R_1 and R_2 are unknown coefficients.

Coupled oscillators

$$F_j = - \sum_{k=1}^n K_{jk} x_k$$

Examples for $n = 2$

$$\mathcal{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\mathcal{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$\mathcal{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

Matrix equation of motion, matrices \mathcal{M} , \mathcal{K} , \mathcal{I} are $n \times n$, vectors \mathcal{X} , \mathcal{Z} are $n \times 1$.

$$\frac{d^2}{dt^2} \mathcal{X}(t) = -\mathcal{M}^{-1} \mathcal{K} \mathcal{X}(t)$$

$$\mathcal{Z}(t) = \mathcal{A} e^{-i\omega t}$$

$$(\mathcal{M}^{-1} \mathcal{K} - \omega^2 \mathcal{I}) \mathcal{A} = 0$$

To obtain the frequencies of normal modes solve:

$$\det(\mathcal{M}^{-1} \mathcal{K} - \omega^2 \mathcal{I}) = 0$$

For $n = 2$

$$\det \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = M_{11} M_{22} - M_{12} M_{21}$$

If the system is driven by force one can find the response amplitudes $\mathcal{C}(\omega_d)$

$$\mathcal{F}(t) = \mathcal{F}_0 e^{-i\omega_d t}$$

$$\mathcal{W}(t) = \mathcal{C}(\omega_d) e^{-i\omega_d t}$$

$$\mathcal{C}(\omega_d) = \begin{bmatrix} c_1(\omega_d) \\ c_2(\omega_d) \end{bmatrix}$$

$$(\mathcal{M}^{-1} \mathcal{K} - \omega_d^2 \mathcal{I}) \mathcal{C}(\omega_d) = \mathcal{F}_0$$

solving the equation above one can find the response amplitudes for the first ($c_1(\omega_d)$) and second ($c_2(\omega_d)$) objects in the system.

Reflection symmetry matrix:

$$\mathcal{S} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Eigenvalues (β) and eigenvectors (\mathcal{A}) of this 2×2 \mathcal{S} matrix:

$$(1) \beta = -1, \mathcal{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(2) \beta = 1, \mathcal{A} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

1D infinite coupled system which satisfy space translation symmetry:

Given a eigenvalue β , the corresponding eigenvector is

$$A_j = \beta^j A_0$$

where

$$A_j(A_0)$$

is the normal amplitude of j th(0th) object in the system.

Consider an one dimensional system which consists infinite number of masses coupled by springs, β can be written as $\beta = e^{ika}$ where k is the wave number and a is the distance between the masses.

Kirchoff's Laws (be careful about the signs!)

$$\text{Node : } \sum_i I_i = 0 \quad \text{Loop : } \sum_i \Delta V_i = 0$$

$$\text{Capacitors : } \Delta V = \frac{Q}{C} \quad \text{Inductors : } \Delta V = -L \frac{dI}{dt} \quad \text{Current : } I = \frac{dQ}{dt}$$

Trigonometric equalities:

$$\sin(a \pm b) = \sin(a) \cos(b) \pm \cos(a) \sin(b)$$

$$\cos(a \pm b) = \cos(a) \cos(b) \mp \sin(a) \sin(b)$$

$$\sin(a) + \sin(b) = 2 \sin\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right)$$

$$\sin(a) - \sin(b) = 2 \cos\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)$$

$$\cos(a) + \cos(b) = 2 \cos\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right)$$

$$\cos(a) - \cos(b) = -2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Some useful integrals involving sin and cos:

$$\frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 1, & \text{if } n = m. \\ 0, & \text{otherwise.} \end{cases}$$

$$\frac{2}{L} \int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 1, & \text{if } n = m. \\ 0, & \text{otherwise.} \end{cases}$$

$$\frac{2}{L} \int_0^L \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0$$

$$\int x \sin(x) dx = \sin(x) - x \cos(x) + C$$

$$\int x \cos(x) dx = \cos(x) + x \sin(x) + C$$

Maxwell Equations in vacuum

$$\begin{aligned}\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{\partial B_z}{\partial t}; & \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{\partial B_x}{\partial t}; & \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{\partial B_y}{\partial t} \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= \mu_0 \epsilon_0 \frac{\partial E_z}{\partial t}; & \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} &= \mu_0 \epsilon_0 \frac{\partial E_x}{\partial t}; & \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} &= \mu_0 \epsilon_0 \frac{\partial E_y}{\partial t} \\ \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} &= 0; & \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} &= 0\end{aligned}$$

Lorentz force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Wave equation for EM fields in vacuum

$$\begin{aligned}\frac{\partial^2 E_i}{\partial x^2} + \frac{\partial^2 E_i}{\partial y^2} + \frac{\partial^2 E_i}{\partial z^2} &= \frac{1}{c^2} \frac{\partial^2 E_i}{\partial t^2} \text{ where } i = x, y, z \\ \frac{\partial^2 B_i}{\partial x^2} + \frac{\partial^2 B_i}{\partial y^2} + \frac{\partial^2 B_i}{\partial z^2} &= \frac{1}{c^2} \frac{\partial^2 B_i}{\partial t^2} \text{ where } i = x, y, z\end{aligned}$$

For EM plane waves in vacuum:

$$\begin{aligned}\vec{B}(\vec{r}, t) &= \frac{1}{c} \hat{k} \times \vec{E}(\vec{r}, t) \\ \vec{E}(\vec{r}, t) &= c \vec{B}(\vec{r}, t) \times \hat{k}\end{aligned}$$

Linear energy density in a string with tension T and mass density ρ_L

$$\frac{dK}{dx} = \frac{1}{2} \rho_L \left(\frac{\partial y}{\partial t} \right)^2 \quad \frac{dU}{dx} = \frac{1}{2} T \left(\frac{\partial y}{\partial x} \right)^2$$

EM energy per unit volume and Poynting vector:

$$U_E = \frac{1}{2} \epsilon_0 \vec{E}^2 \quad U_B = \frac{1}{2\mu_0} \vec{B}^2 \quad \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

Transmission and reflection

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \quad T = \frac{2Z_1}{Z_1 + Z_2}$$

Phase velocity and impedance:

$$v = \sqrt{\frac{T}{\rho_L}}, \quad Z = \sqrt{T\rho_L} \quad (\text{string})$$

$$v = \sqrt{\frac{1}{LC}}, \quad Z = \sqrt{\frac{L}{C}} \quad (\text{transmission line})$$

Snell's law

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

Fourier transform

$$f(t) = \int_{-\infty}^{\infty} d\omega C(\omega) e^{-i\omega t}$$

$$C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t}$$

Delta function

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega - \omega')t} dt = \delta(\omega - \omega')$$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

$$\int_{-\infty}^{\infty} \delta(x - a) f(x) dx = f(a)$$

Electric and magnetic field from an accelerated charge:

$$\vec{E}(\vec{r}, t) = -\frac{qa_{\perp}^{\vec{r}}(t - |\vec{r}|/c)}{4\pi\epsilon_0 r c^2}$$

$$\vec{E}(\vec{r}, t) = \frac{\hat{r} \times \vec{B}}{c}$$

Total power emitted by the accelerated charge:

$$P(t) = \frac{q^2 a^2(t - r/c)}{6\pi\epsilon_0 c^3}$$

Interference of two sources with amplitudes A_1 and A_2 with a relative phase difference δ :

$$\langle I \rangle \propto (A_1^2 + A_2^2 + 2A_1 A_2 \cos \delta)$$

Interference of N fields of equal amplitude with phases $\delta_{m+1} - \delta_m = \delta$:

$$\langle I \rangle = \langle I_0 \rangle \left[\frac{\sin(N\delta/2)}{\sin(\delta/2)} \right]^2$$

Single slit diffraction where β is the phase difference between rays coming from edges and the center of the slit:

$$\langle I \rangle = \langle I_0 \rangle \left[\frac{\sin(\beta)}{\beta} \right]^2$$

Rayleigh's criterion for resolution: Diffraction peak of one image falls on the first minimum of the diffraction pattern of the second image.

Electric field transmission and reflection ratios, magnitude and sign, for radiation incident normally on an interface between lossless dielectrics with indices of refraction n_1 and n_2 .

$$\frac{E_t}{E_i} = \frac{2n_1}{n_1 + n_2} \quad \frac{E_r}{E_i} = \frac{n_1 - n_2}{n_1 + n_2}$$

Schrodinger's Equation

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x, t) \right] \psi(x, t)$$

where V is the potential energy, m is the mass of the particle and ψ is the wave function.

MIT OpenCourseWare
<https://ocw.mit.edu>

8.03SC Physics III: Vibrations and Waves
Fall 2016

For information about citing these materials or our Terms of Use, visit: <https://ocw.mit.edu/terms>.