Formula Sheet

The differential equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f\cos(\omega t + \phi) \tag{1}$$

Has the general solutions;

$$\frac{\gamma}{2} < \omega_0 : \quad X(t) = A_1 e^{-\left(\frac{\gamma}{2}\right)t}\cos(\omega' t + \beta) + X_p(t) \tag{2}$$

$$\frac{\gamma}{2} = \omega_0 : \quad X(t) = (A_1 + A_2 t)\, e^{-\left(\frac{\gamma}{2}\right)t} + X_p(t) \tag{3}$$

$$\frac{\gamma}{2} > \omega_0 : \quad X(t) = A_1 e^{-\Gamma_+ t} + A_2 e^{-\Gamma_- t} + X_p(t) \tag{4}$$

with

$$X_p(t) = A(\omega)\cos(\omega t - \delta(\omega) + \phi) \tag{5}$$

and

$$\omega' = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \qquad \Gamma_\pm = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \tag{6}$$

$$A(\omega) = f/\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2} \qquad \tan(\delta(\omega)) = \gamma\omega/(\omega_0^2 - \omega^2) \tag{7}$$

Idealized relations for voltage/emf across circuit elements:

1. Capacitor: $V_C = \dfrac{Q}{C}$
2. Resistor: $V_R = IR$
3. Self Inductance: $V_L = L\dfrac{dI}{dt}$

Classical Scalar Wave Equation in 3-D: $\dfrac{\partial^2\Psi}{\partial t^2} = v^2\nabla^2\Psi$

The plane wave solution is: $\Psi(\vec{r},t) = A\cos(\vec{k}\cdot\vec{r} \pm \omega t + \phi)$

The spherical wave solution is: $\Psi(\vec{r},t) = A\dfrac{\cos(kr \pm \omega t + \phi)}{r}$

Classical Wave Equation in 1-D: $\dfrac{\partial^2\Psi}{\partial t^2} = v^2\dfrac{\partial^2\Psi}{\partial x^2}$

The standing wave solution is: $\Psi(x,t) = A\cos(\frac{\omega}{v}x + \phi_x)\cos(\omega t + \phi_t)$

The progressive wave solution is: $\Psi(x,t) = f(t \pm \frac{x}{v})$

For String: $v = \sqrt{\dfrac{T}{\mu}}$ $\quad Z = \sqrt{T\mu}$ $\quad \langle P\rangle = \dfrac{1}{2}\dfrac{F_0^2}{Z}$

For Sound: $v = \sqrt{\dfrac{\kappa}{\rho}}$ $\quad Z = \sqrt{\kappa\rho}$ $\quad \kappa = -V\dfrac{\partial P}{\partial V}$

For Torsion: $v = \sqrt{\dfrac{k}{I}}$ $\quad Z = \sqrt{kI}$

For Transmission Line: $v = \dfrac{1}{\sqrt{LC}}$ $\quad Z = \sqrt{\dfrac{L}{C}}$ $\quad \langle P\rangle = \dfrac{1}{2}\dfrac{V_0^2}{Z}$

$$v_{phase} = \frac{\omega}{k} = \nu\lambda \qquad v_{group} = \frac{\partial\omega}{\partial k}$$

For a displacement wave on a string: $T = \dfrac{2Z_1}{Z_1 + Z_2}$ $\quad R = \dfrac{Z_1 - Z_2}{Z_1 + Z_2}$

For a voltage wave on a transmission line: $T = \dfrac{2Z_2}{Z_1 + Z_2}$ $\quad R = \dfrac{Z_2 - Z_1}{Z_1 + Z_2}$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad U_E = \frac{1}{2}\epsilon_0 E^2$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad U_B = \frac{1}{2\mu_0} B^2$$

$$\vec{\nabla} \times \vec{B} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J} \qquad \vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

Electromagnetic wave in vacuum: $\dfrac{\partial^2 \vec{E}}{\partial t^2} = \dfrac{1}{\mu_0 \epsilon_0} \nabla^2 \vec{E}$ $\quad$ Radiation Pressure: $\dfrac{\vec{S}}{c}$

For a progressive wave solution in vacuum: $\dfrac{|E|}{|B|} = c = \dfrac{1}{\sqrt{\mu_0 \epsilon_0}}$

Radiation due to the acceleration of charge:

$$\vec{E}(\vec{r}, t) = -\frac{q\vec{a}_\perp(t')}{4\pi\epsilon_0 r c^2} \qquad \vec{B} = \frac{\hat{r} \times \vec{E}}{c} \qquad t' = t - \frac{|r|}{c}$$

Total radiated power from accelerated charge (Larmor formula): $P(t) = \dfrac{q^2 a^2(t')}{6\pi\epsilon_0 c^3}$

Boundary conditions at the surface of a perfect conductor (for time-varying fields): $E_\parallel = 0$ and $B_\perp = 0$.

For most dielectrics ($K_M \approx 1$): $n = \sqrt{K_E K_M} \approx \sqrt{K_E}$ $\quad v_{phase} = \dfrac{c}{n}$

Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$

Reflection and transmission of electromagnetic waves at normal incidence:

$$E_{reflected} = E_{incident} \frac{n_1 - n_2}{n_1 + n_2}$$
$$E_{transmitted} = E_{incident} \frac{2n_1}{n_1 + n_2}$$

For interference from N slits where a separation d between two slits,

$$I(\theta) = I_0 \frac{\sin^2(\frac{N\pi}{\lambda} d \sin\theta)}{\sin^2(\frac{\pi}{\lambda} d \sin\theta)}$$

Diffraction intensity from a slit of width D,

$$I(\theta) = I_0 \frac{\sin^2(\frac{\pi}{\lambda} D \sin\theta)}{(\frac{\pi}{\lambda} D \sin\theta)^2}$$

Rayleigh's criterion for resolution: Diffraction peak of one images falls on the first minimum of the diffraction pattern of the second image.

For a periodic function with period Λ

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(n\frac{2\pi x}{\Lambda}\right) + \sum_{n=1}^{\infty} B_n \sin\left(n\frac{2\pi x}{\Lambda}\right)$$

with

$$A_0 = \frac{2}{\Lambda}\int_0^{\Lambda} f(x)dx \qquad A_n = \frac{2}{\Lambda}\int_0^{\Lambda} f(x)\cos\left(n\frac{2\pi x}{\Lambda}\right)dx \qquad B_n = \frac{2}{\Lambda}\int_0^{\Lambda} f(x)\sin\left(n\frac{2\pi x}{\Lambda}\right)dx$$

Trigonometric identities:

$$\begin{aligned}
\sin(A+B) &= \sin A\cos B + \cos A\sin B \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B \\
\sin A + \sin B &= 2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right] \\
\cos A + \cos B &= 2\cos\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right] \\
\sin A - \sin B &= 2\cos\left[\frac{1}{2}(A+B)\right]\sin\left[\frac{1}{2}(A-B)\right] \\
\cos A - \cos B &= -2\sin\left[\frac{1}{2}(A+B)\right]\sin\left[\frac{1}{2}(A-B)\right] \\
e^{j\theta} &= \cos\theta + j\sin\theta
\end{aligned}$$

Some useful integrals:

$$\begin{aligned}
\int_0^L \sin(\frac{n\pi x}{L})\sin\frac{m\pi x}{L})dx &= \begin{cases} \frac{L}{2} & \text{for } m=n \\ 0 & \text{for } m\neq n \end{cases} \\
\int_0^L \cos(\frac{n\pi x}{L})\cos\frac{m\pi x}{L})dx &= \begin{cases} \frac{L}{2} & \text{for } m=n \\ 0 & \text{for } m\neq n \end{cases} \\
\int x\sin(ax)dx &= \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C \\
\int x\cos(ax)dx &= \frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C
\end{aligned}$$

$$
\begin{aligned}
\text{div } \vec{A} &= \vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\text{curl } \vec{A} &= \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}
\end{aligned}
$$

$$
\nabla^2 \vec{A} = \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} \right) \hat{x} + \left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} \right) \hat{y} + \left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} \right) \hat{z}
$$

Physical constants:

$$
\begin{aligned}
e &= 2.718 \\
e^{-1} &= 0.3679 \\
\pi &= 3.1416 \\
c &= 3.0 \times 10^8 \text{ m/s}
\end{aligned}
$$

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
PHYSICS DEPARTMENT

Physics 8.03: Vibrations and Waves

Exam 1

FAMILY (Last) Name

GIVEN (First) Name

Student ID Number

Recitation Section:
(check one)

☐	R01	TR	10	Prof. Jarillo-Herrero
☐	R02	TR	11	Prof. Jarillo-Herrero
☐	R03	TR	1	Prof. Weinberg
☐	R04	TR	2	Prof. Weinberg

Instructions:

1. Do not remove any pages of the exam, except the formula sheet.
2. This is a closed book exam.
3. Do all SIX (6) problems.
4. SHOW ALL WORK. Print your name on each sheet.
5. CALCULATORS, BOOKS, COMPUTERS and CELL PHONE are **NOT ALLOWED**.

Points:

Problem	Maximum	Score	Grader
Problem 1:	16		
Problem 2:	16		
Problem 3:	16		
Problem 4:	16		
Problem 5:	18		
Problem 6:	18		

Problem 1: 16 Points

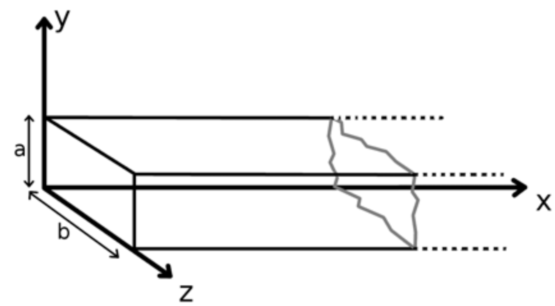


Figure 1: A perfectly conducting waveguide.

The electric field for a TE mode in an infinitely long (in the x direction) perfectly conducting rectangular waveguide ($a < b$) is given by;

$$\vec{E}(x, y, z, t) = E_0 \cos(k_y y + \phi_y) \cos(k_x x - \omega t)\hat{z} \tag{8}$$

(1.a)

(4pts) Find k_y and ϕ_y that satisfy the boundary conditions.

(1.b)

(4pts) Write down the dispersion relation for this mode of the waveguide.

(1.c)

(4pts) What is the lowest frequency that will propagate in this mode?

(1.d)

(4pts) What is the magnetic field $\vec{B}(x, y, z, t)$ associated with the electric field of this mode?

Problem 2: 16 Points

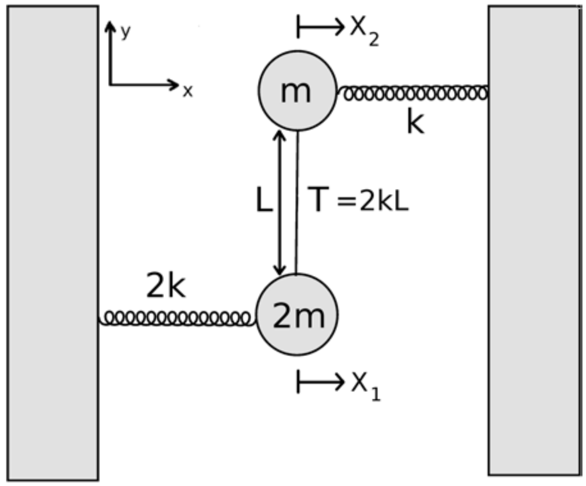


Figure 2: A system of coupled oscillators.

The figure above shows a system of masses. The mass of $2m$ is connected to an immobile wall with a spring of constant $2k$, while the mass of m is connected to an immobile wall with a spring of constant k. The masses are then coupled to each other with an elastic band of length L, under tension $T = 2kL$. The masses are constrained to move in the x direction only. At equilbrium the masses have the same x position and the springs are uncompressed. There is no friction or gravity. The displacements from equilibrium are small enough ($x_1, x_2 \ll L$), so that the tension in the band stays constant.

(2.a)

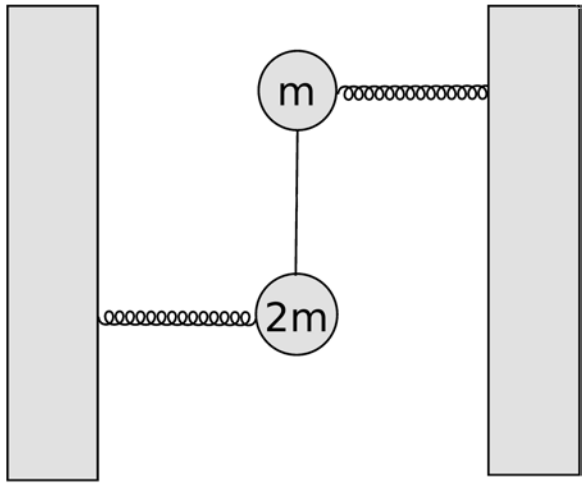
(5pts) Write down the coupled differential equations describing the displacement of the masses from equilibrium $\{x_1, x_2\}$.

(2.b)

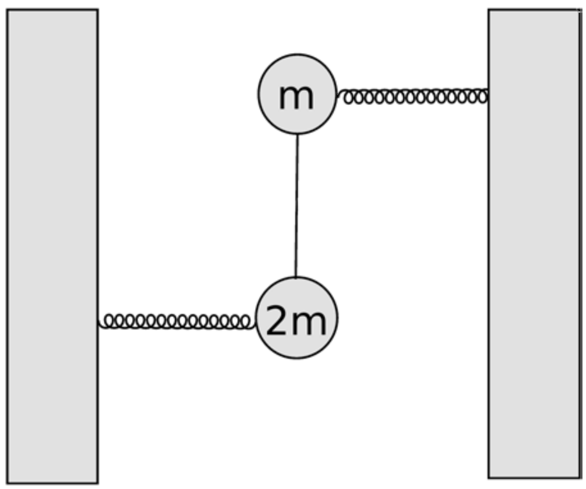
(7pts) Find the normal mode frequencies of the system.

(2.c)

(4pts) On the two figures included on the next page sketch the normal modes of the system, be sure to clearly indicate both the magnitude and direction of the motion of the masses.



m
2m



m
2m

Problem 3: 16 Points

(3.a)

(5pts) An optical fiber consists of a solid rod of material with index of refraction n_f surrounded by a cylindrical shell of material with index n_c. Find the largest angle θ so that a wave incident on the solid rod from air with index n_a remains in the solid rod (express your answer in terms of n_f, n_c, and n_a).

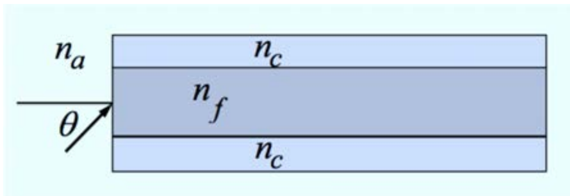


Figure 3: An optical fiber.

(3.b)

(4pts) Unpolarized light propagating in vacuum reflects off the surface of a liquid with index n. The reflected ray strikes a screen 25cm away at a height of 20cm and is observed to be 100% polarized. What is is n?

(3.c)

(7pts) Consider a medium in which waves propagate with a dispersion relation

$$\omega^2 = \omega_0^2 + A^2 k^2 \tag{9}$$

where ω is the wave (angular) frequency, k is the wave number, and ω_0 and A are real constants.

(i) What is the range of frequencies ω for which waves can propagate?

(ii) Compute v_{phase} and v_{group}. Make a carefully labeled sketch of each as a function of ω in the plots below.

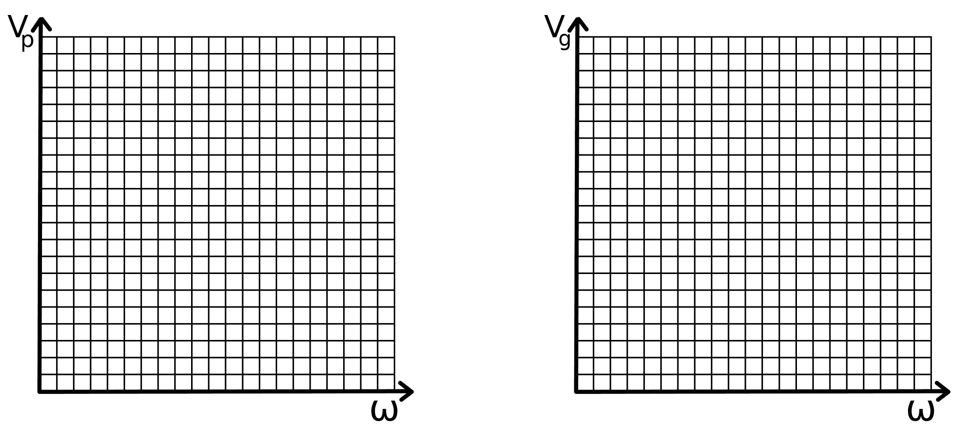


Figure 4: Plot the phase/group velocity

Problem 4: 16 Points

A monochromatic beam is incident on N slits, which results in a intensity pattern as a function of angle on a screen some distance away as shown in the figure below. Each slit has a width D and the distance between the centers of the slits is d. The distance between the screen and the slits is very large.

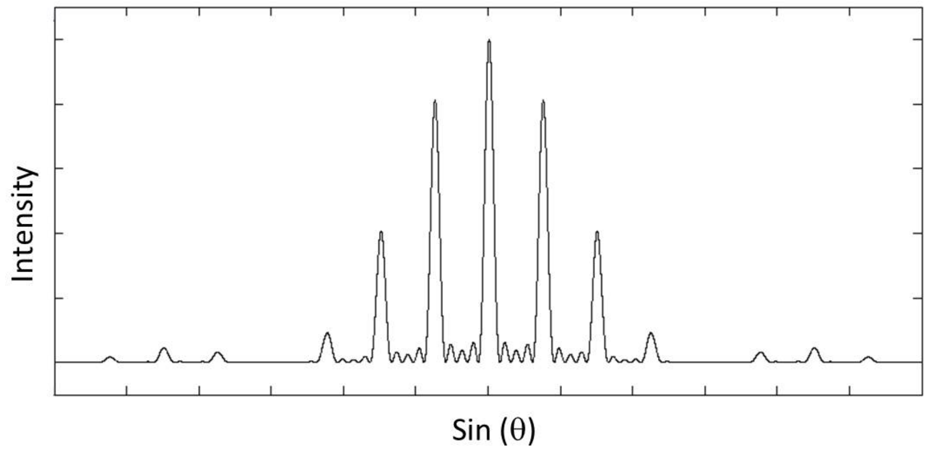


Figure 5: Interference pattern due to N slits.

From the pattern deduce the following:

(4.a)

(6pts) The number of slits N on which the beam is incident. Explain your reasoning.

(4.b)

(6pts) The ratio d/D. Explain your reasoning.

(4.c)

(4pts) Now suppose that the width of the slits is reduced to ~ 0, while the intensity of the monochromatic beam is increased so that the intensity of the central maximum is unchanged. On top of the plot (showing the original intensity pattern in dashed lines) on the next page, draw the resulting intensity pattern.

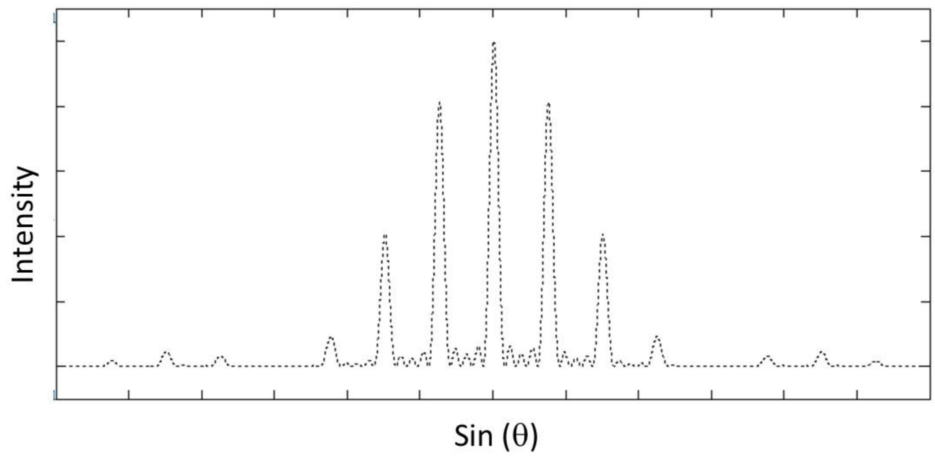


Figure 6: Plot the resulting intensity pattern as $D \to 0$.

Problem 5: 20 Points

A string of length $2L$ with mass density μ is placed under tension T and is fixed at both ends. At time $t = 0$, the displacement of the string is zero everywhere but it is struck so that a transverse velocity is imparted to a section of the string. The intial conditions of the string are ($a \ll L$);

$$y(x, t = 0) = 0 \tag{10}$$

$$\dot{y}(x, t = 0) = \begin{cases} v_0 & : \quad L - a \leq x < L \\ -v_0 & : \quad L \leq x < L + a \\ 0 & : \quad \text{elsewhere} \end{cases} \tag{11}$$

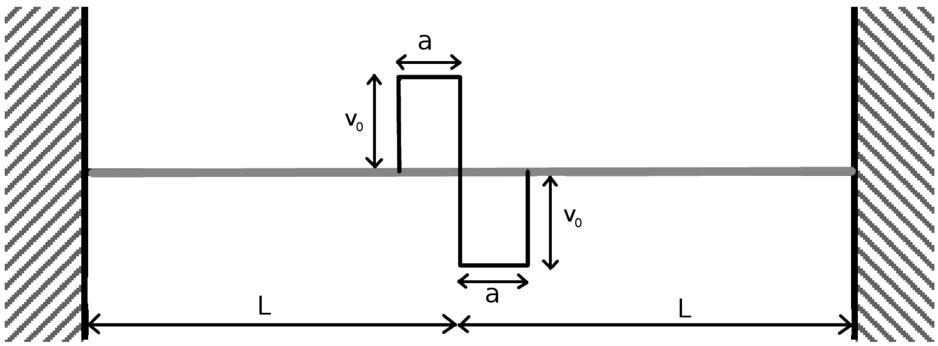


Figure 7: The intial transverse velocity of the string at time $t = 0$. The initial displacement is zero everywhere.

(5.a)

(3pts) Using the plot (Figure #8) provided on the next page, sketch the first three normal modes of vibration for this string, regardless of whether or not they are excited.

(5.b)

(10pts) What is the amplitude of the n-th normal mode after the string is struck? What is the lowest unexcited mode?

(Problem continues on the next page.)

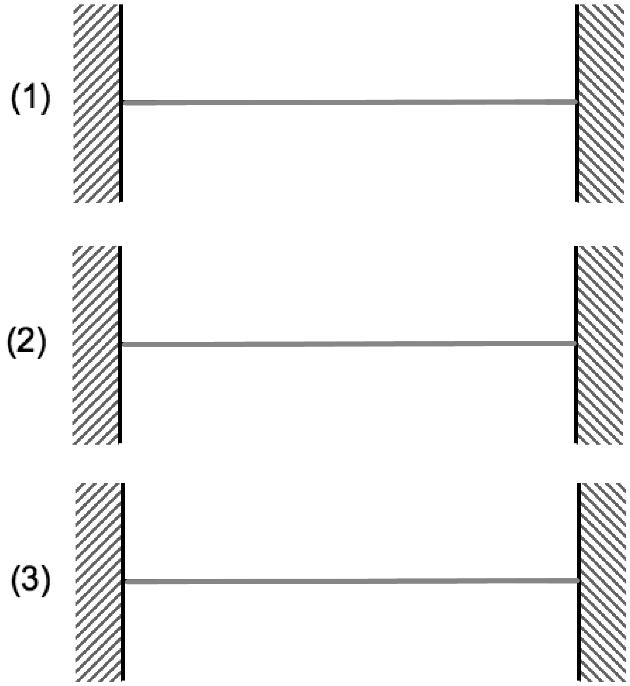


Figure 8: Plot the first three normal modes.

(5.c)

(5pts) Sketch the *displacement* of the string at time $t = \frac{L}{2}\sqrt{\frac{\mu}{T}}$ in the plot below.

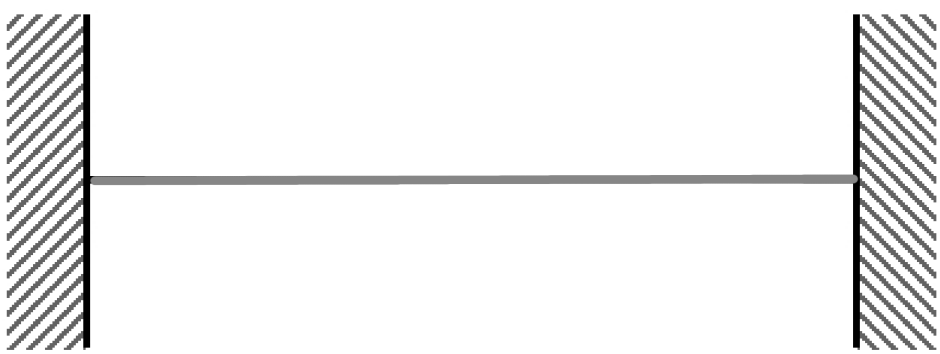


Figure 9: Plot the displacement of the string.

Problem 6: 18 Points

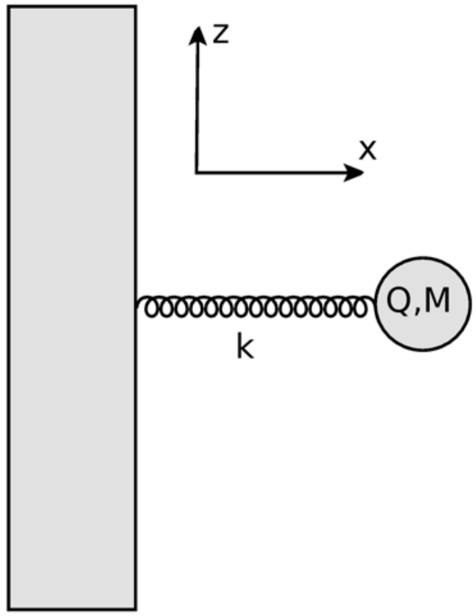


Figure 10: An oscillating charge.

A charged particle of mass M and charge $+Q$ is attached to the end of a spring of spring constant k. The spring lies along the x-axis and the equilibrium point is at the origin. The particle is displaced from equilibrium by a distance A in the x direction, and released at $t = 0$. Assume that the size of the particle is much smaller than A, so it can be treated as a point charge and that the damping rate is very small.

(6.a)

(4pts) Calculate the electric field radiated by the particle along an arbitrary direction in the $x - z$ plane, at a distance R, where $R \gg A$.

(6.b)

(4pts) Calculate the total time averaged power radiated by the particle.

(6.c)

(6pts) Assuming that the power radiated does not change appreciably as a function of time, give a simple rough estimate of the time it will take for the particle to decrease its amplitude of oscillation to $1/e$ of its initial value. Is this assumption realistic?

(6.d)

(4pts) A more refined estimate can be obtained using that $dA/dt = (dA/dE) \times (dE/dt)$, and using the average power radiated over a given cycle for dE/dt. Use this to calculate the time it will take the particle to decrease its oscillation amplitude to $1/e$ of its initial value.