PROFESSOR: Any questions from last lecture from before spring break? No questions, nothing? Nothing at all? We you all still on vacation? Thank you. Yeah.

AUDIENCE: Could you talk about coherence states?

PROFESSOR: Ah. OK, well, that's a great question. So coherence states-- actually, how many people have looked at the optional problems? Nice, OK, good. Good, so coherence states were a topic that we touched on the problem sets and on the optional problems, and the optional problems are mostly on the harmonic oscillator and nice problems revealing some of the structure of the harmonic oscillator, and it generalizes quite boldly.

But here's the basic idea of coherence states. Let me just talk you through the basic ideas rather than do any calculations. So what's the ground state of a harmonic oscillator? What does its wave function look like?

AUDIENCE: Gaussian.

PROFESSOR: It's a Gaussian, exactly. It's minimum uncertainty wave packet. How does it evolve in time?

AUDIENCE: Phase.

PROFESSOR: Yeah, phase. It's an energy eigenstate, it's the ground state, so it just evolves in time with the phase. So if we look at the wave function for the ground state, phi naught, it's something like e to the minus x squared over 2a squared with some normalization coefficient, which I'm not going to worry about. So this is a minimal uncertainty wave packet. Its position distribution is time independent, because it's a stationary statement. It's momentum distribution, which is also a Gaussian, the

Fourier transform, is time independent. And so this thing up to its rotation by the overall phase just sits there and remains a Gaussian.

Now, here's a question. Suppose I take my harmonic oscillator potential, and I take my Gaussian, but I displace it a little bit. It's the same ground state, it's the same state but I've just displaced it over a little. What do you expect to happen? How will this state evolve in time? So, we know how to solve that problem. We take this state, it's a known wave function at time 0. We expand it in the basis of energy eigenstates, each energy eigenstate evolves in time with a phase, so we put in that phase and redo the sum, and recover the time evolution of the full state. And we've done this a number of time on the problem sets. Yeah.

AUDIENCE: Would the Gaussian, the displaced Gaussian evolve the same way, keep its width, if it had any other initial width other than the one of the ground state?

PROFESSOR: Let me come back to that, because it's a little more of a precise question. So we know how to solve this problem practically, algorithmically. But here's a nice fact. I'm not going to derive any equations, that's part of the point of the optional problems, but here's a nice fact about this state. So it's clearly a minimum uncertainty wave packet because at time 0, because it's just the same Gaussian just translated over a little bit.

So what we'd expect, naively, from solving is we expand this in Fourier modes and then we let this the system evolve in time. We let each individual Fourier mode-- or, sorry, Fourier mode-- we let each individual energy eigenstate evolve in time, pick up a phase, and now what we get is a superposition. Instead of sum over n at time 0, we'd have sum over n cn phi n of x. Now, as a function of time, we get e minus I omega nt, and these phases are going to change the interference from summing up all these energy eigenstates, and so the system will change in time. Because the way that the various terms in the superposition interfere will change in time.

So very naively, what you might expect, if you just took a random function-- for example, if I took a harmonic oscillator potential and I took some stupid function that did something like this. What would you expect it to do over time? Well, due to all the complicated interference effects, you'd expect this to turn into, basically, some schmutz. Just some crazy interference pattern.

The thing that's nice about a coherent state, and this is where it gets its name, is the way that all of these interference effects conspire together to evolve the state is to leave it a Gaussian that does nothing but translates in time. A coherent state is coherent because it remains coherently a Gaussian as it moves along. It oscillates back and forth. And in fact, the peak, the center of this Gaussian wave packet, oscillates with precisely the frequency of the trap. It behaves just as a classical particle would have had you displaced it from the center of the harmonic oscillator trap, it oscillates back and forth.

Now, on the other hand, you know that its momentum is changing in time, right. And at any given point, as you've shown on the problem set that's due tomorrow-- or you will have shown-- at any given moment in time, the momentum can be understood as the overall phase. You could just change that momentum by the overall phase, so it's the spatial rate of change of the phase, e to the ikx.

So you know that the way the phase depends on position is changing over time. So it can't be quite so simple that the wave function, rather than the probability distribution, is remaining a Gaussian over time. It's not, it's got all sorts of complicated phases. But the upshot is that the probability distribution oscillates back and forth perfectly coherently. So that seems a little magical, and there's a nice way to understand how that magic arises, and it's to understand the following. If I take a state, phi 0, but I translate it x minus x0. OK.

So we know that this state, phi 0 of x, was the ground state, what does it mean to be the ground state of the harmonic oscillator? How do I check if I'm the ground state of the harmonic oscillator? Look, I walk up to you, I'm like, hey, I'm the ground state of the harmonic oscillator. You're suspicious. What do you do?

AUDIENCE: Annihilate.

PROFESSOR: Annihilate. Exactly. You act with the annihilation operator. Curse you. So you act with the annihilation operator and you get 0, right. What happens if I act with the annihilation operator on this? Is this the ground state? No, it's been displaced by x0. And meanwhile, I told you that it oscillates back and forth. So what happens if you act with the annihilation operator? Should you get 0? No, what are you going to get?

AUDIENCE: Something weird.

PROFESSOR: Some random schmutz, right? If you just take a and you act on some stupid state, you'll just get some other stupid state. Except for Gaussians that have been displaced, you get a constant times the same wave function, phi 0 of x minus x0.

Aha. It turns out that these displaced Gaussians are eigenstates of the annihilation operator. What does that mean? Well, it means they're coherent states. And so the optional problems are a working through of the study of the eigenfunctions of the annihilation operator, the coherent states. Is that cool? It's a state, it's a superposition. You can think about it like this. It's a superposition of the energy eigenstate. Any state is a superposition of energy eigenstates, and a coherent state is just some particularly special superposition of energy eigenstates. So you can think about it literally as c0 phi 0 plus c1 phi 1 plus c2 phi 2, blah blah blah.

What does the annihilation operator do, what does the lowering operator do? It takes any state and then lowers it with some coefficient. So a coherent state, an eigenstate of the annihilation operator, must be a state such that if you take c 0 phi 0 plus c1 phi 1 plus c2 phi 2 and you hit it with the annihilation operator, which will give you see c1 phi 0 plus c2 phi 1 divided by root 2 plus dot, dot, dot-- or times root 2 plus dot, dot, dot. It gives you the same state back.

So that gave you a relation between the coefficients c0 and c1, c1 and c2, c2 and c3. They've all got to be suitable multiples of each other. And what you can show-- and this is something you work through, not quite in this order, on the optional problems-- what you can show is that doing so it gives you a translated Gaussian. OK.

So physically, that's what a coherent state is. Another way to say what a coherent state is, it's as close to a classical object as you're going to get by building a quantum mechanical wave function. It's something that behaves just like a classical particle in that potential would have behaved, in the harmonic oscillator potential. Did that answer your question? OK. Anything else?

So, Matt works with these for a living. Matt, do you want to add anything? All right. I like these. They show up all over the place. This is like the spherical cow of wave functions because it's awesome. It's an eigenfunction, it's an annihilation operator, it behaves like a classical particle.

AUDIENCE: Is there an easy way to define coherent states in potentials where you don't have nice operators like a?

PROFESSOR: No, I mean, that's usually what we mean by a coherent state. So the term coherent state is often used interchangeably to mean many different things, which in the case of the harmonic oscillator, are identical. One is a state which is a Gaussian. OK. So in the harmonic oscillator system, that's a particularly nice state because it oscillates particularly nicely and it maintains its probability distribution as a function of time roughly by translating, possibly, some phases.

But people often use the phrase coherent state even when you're not harmonic. And it's useful to keep in mind that we're often not harmonic, nothing's truly harmonic. You had another question a moment ago though, and your question was about the width. Yeah.

AUDIENCE: It just translates without changing its width, with or without changing its shape at all. If the Gaussian were of a different width, would that still happen?

PROFESSOR: Yeah, it does, although the details of how it does so are slightly different. That's called a squeeze state. The basic idea of a squeeze state is this. Suppose I have a harmonic oscillator-- and this is actually one way people build squeeze states in labs-- so suppose I have a harmonic oscillator and I put the system in a ground state. So there's its ground state. And its width is correlated with the frequency of the potential.

Now, suppose I take this potential at some moment in time and I control the potential, the potential is created by some laser field, for example. And some annoying grad student walks over to the control panel and doubles the power of the laser. All of a sudden, I've squeezed the potential. But my system is already in a state which is a Gaussian, it's just the wrong Gaussian. So what does this guy do?

Well, this is funny. The true ground state, once we've squeezed, the true ground state would be something that's much narrower in position space. I didn't draw that very well, but it would be much narrower in position space, and thus its distribution would be much broader in momentum space. So the state that we put the system, or we've left the system in, has too much position uncertainty and too little momentum uncertainty.

It's been squeezed compared to the normal state in the delta x delta p plane. It's uncertainty relation is still extreme, it still saturates the uncertainty bound because it's a Gaussian, but it doesn't have the specific delta x and delta p associated with the true ground state of the squeezed potential.

OK, so now you can ask what does this guy do in time. And that's one of the optional problems, too, and it has many of the nice properties of coherent states, it's periodic, it evolves much like a classical particle, but it's uncertainties are different and they change shape. It's an interesting story, that's the squeezed state. Yeah.

AUDIENCE: Do other potentials have coherent states that are Gaussian?

PROFESSOR: It sort of depends. So other potentials have states that behave classically. Yeah. There are generally-- so systems that aren't the harmonic oscillator do have very special states that behave like a classical particle. But they're not as simple as annihilations by the annihilation operator. Come to my office and I'll tell you about analogous toys for something called supersymmetric quantum models, where there's a nice story there. OK.

I'm going to cut off coherent states for the moment and move on to where we are now. OK, so last time we talked about scattering of a particle, a quantum particle of mass m against a barrier. And we made a classical prediction, which I didn't quite phrase this way. But we made a classical prediction that if you took this particle of mass m and threw it against a barrier of height v0, that the probability that it will transmit across this barrier to infinity is basically 0. So this is the classical prediction. It will not transmit to infinity until the energy goes above the potential. And when the energy is above the potential, it will slow down, but it will transmit 100% of the time.

So this is our classical prediction. And so we sought to solve this problem, we did. It was pretty straightforward. And the energy eigenstate took the form-- well, we know how to solve it out here, we know how to solve it out here, because these are just constant potentials so it's just plain waves. Let's take the case of the energy is less than the potential. So this guy.

Then out here, it's in a classically disallowed state, so it's got to be a exponential. If we want it to be normalized, well, we need it to be a decaying exponential. Out here it's oscillatory, because it's a classically allowed region. And so the general form of the wave function of the energy eigenstate is a superposition of a wave moving this way with positive momentum, a wave with negative momentum, a contribution with minus k, and out here it had to be the decaying exponential.

And then by matching, requiring that the wave function was oscillatory and then exponentially decaying out here, and requiring that it was continuous and differentiably continuous-- that its derivative was continuous-- we found matching conditions between the various coefficients, and this was the solution in general. OK.

Now, in particular, this term, which corresponds to the component of the wave function moving towards the barrier on the left hand side, has amplitude 1. This term, which corresponds to a wave on the left with negative momentum, so moving to the left, has amplitude k minus i alpha over k plus i alpha, where h bar squared k squared upon 2m is equal to-- that's just the energy, e-- and h bar squared alpha squared over 2m-- that's on this side-- is equal to v0 minus e.

But this, notably, is a pure phase. And we understood that, so we'll call this parameter r, because this is the reflected wave and this is an amplitude rather than a probability, so we'll call it little r. And we notice that the probability that we get out-- good. So if we want to ask the question, what is the transmission probability, the probability that I get from far out here to far out here, what is that probability?

So if I consider a state that starts out as a localized wave packet way out here and I send it in but with energy below the barrier, what's the probability that I'll get arbitrarily far out here, that I will subsequently find the particle very far out here? 0, right, exactly. And you can see that because here's the probability amplitude, the norm squared is the probability density as a function of the position. And it goes like a constant times e to the minus alpha x. For large x, this exponential kills us.

And we made that more precise by talking about the current. We said look, the current that gets transmitted is equal to-- that's a funny way to write things. H bar-- sorry. The current that gets transmitted, which equal to h bar upon 2mi psi complex conjugate dx psi minus psi dx psi complex conjugate on the right. So we'll put right, right, right, right.

This is just equal to 0, and the easiest way to see that is something you already showed on a problem set. The current vanishes when the wave function can be made real up to a phase. And since this is some number, but in particular it's an overall constant complex number, and this is a real wave function, we know that this is 0. So the transmitted current out here is 0. The flux of particles moving out to the right is 0.

So nothing gets out to the right. So that was for the energy less than the potential, and we've now re-derived this result, the classical prediction. So the classical prediction worked pretty well. Now, importantly, in general, we also wanted to define the transmission probability a little more carefully, and I'm going to define the transmission probability as the current of transmitted particles on the right divided by the current of probability that was incident. And similarly, we can define a probability that the particle reflects, which is the ratio of the reflected current, the current of the reflected beam, to the incident current.

So this is going to be the transmission probability and the reflection probability. Yeah.

AUDIENCE: Is there a square on t?

PROFESSOR: Thank you. OK. So let's do a slightly different example. In this example, I want to study the same system, the wall, but I want to consider a ball incident from the left-- a ball. An object, a quantum particle incident from the left, with energy greater than v. Greater than v0. So again, classically, what do you expect in this case. You send in the particle, it loses a little bit of energy going up the potential barrier, but it's still got a positive kinetic energy, and so it just keeps rolling. Just like my car making it up the driveway, just barely makes it.

OK. So here again, what's the form of the wave function. Phi e can be put in the form on the left. It's going to be e to the ikx plus b to the minus ikx, and that's over here on the left. And on the right it's going in the form ce to the ik-- in fact, let me call this k1x and k2x-- or sorry, k1x, which is on the left. H bar squared k1 squared upon 2m is equal to e. And on the right, we're going to have k2x because it's, again, a classically allowed region, it's going to be oscillatory but with a different momentum. e to the ik2x plus de to the minus ik2x where h bar squared k2 squared upon 2m is equal to e minus v0, which is a positive number, so that's good. So here's our wave function.

Now, before we do anything else, I want to just interpret this quickly. So again, this is like a wave that has positive momentum, and I'm going to say that it's like a contribution, a term in the superposition, that is moving to the right, and the way to think of it as moving to the right, well, first off, it's got positive momentum. But more to the point, this is an energy eigenstate. So how does this state evolve in time? It's an energy eigenstate with energy, how does it evolve in time.

AUDIENCE: [INAUDIBLE]

PROFESSOR: It rotates by an overall phase, exactly. So we just get this times an e to the minus i omega t, where h bar omega is equal to e, the energy. And that's true on the left or right, because the energy is just a constant. So this term goes as e to the ikx minus i omega t, or kx minus omega t. And this is a point of constant phase in this move to the right. As t increases-- in order for the phase to be constant or for the phase, for example, to be 0-- as t increases, x must increase. By assumption here, k is a positive number.

On the other hand, this guy-- that was a k1-- the second term is of the form e to the minus ik1x, and when I add the omega t, minus i omega t, this becomes kx plus omega t. So point of constant phase, for example, phase equals 0, in order for this to stay 0 then, as t increases, x must become negative. x must decrease. So that's why we say this corresponds to a component of the wave function, a term in the superposition which is moving to the left. And this is one which is moving to the right. Sorry, my right, your left. Sorry, this is moving to your right and this is moving to your left. OK? It's that little z2, it's harder than it seems. OK.

In this set up, we could imagine two different kinds of scattering experiments. We can imagine a scattering experiment where we send in a particle from the left and ask what happens. So if you send in a particle from the left, what are the logical possibilities?

AUDIENCE: It can go through.

PROFESSOR: It could go through, or?

AUDIENCE: Reflect back.

PROFESSOR: Reflect back. Are you ever going to get a particle spontaneously coming from infinity out here? Not so much. So if you're sending a particle in from the left, what can you say about these coefficients? d equals 0 and a is not 0, right. Because d corresponds to a particle on the right hand side coming in this way. So that's a different scattering experiment. So in particular, coming in from the left means d equals 0 and a not equal to 0. Coming in from the right means a equals 0 and d not equal to 0 by exactly the same token. Cool?

So I want to emphasize this. a and d, when you think of this as a scattering process, a and d are in guys. In, in. And c and b are out. This term corresponds to moving away from the barrier, this term corresponds to moving away from the barrier, just on the left or on the right. OK? Out and out. Everyone happy with that? Yeah.

AUDIENCE: According to that graph, if d is-- no, that one. Yeah. If d is bigger than [INAUDIBLE] then the transplants will be 1. Doesn't that mean that d is also 0?

PROFESSOR: So this is the classical prediction. I've written it, classical prediction. So let me rephrase. The question was basically, look, that classical prediction implies that b must be 0 on the other scattering process, and that doesn't sound right. So is that true? No, it's not true, and we'll see it again in a second. So very good intuition. That was good. OK, good.

So in general, if we have a general wave function, general superposition of the two states with energy e with a,b,c, and d all non-0, that's fine. That just corresponds to sending some stuff in from the right and some stuff in-- sorry, some stuff in from the left and some stuff in from the right. Yeah? But it's, of course, going to be easier if we can just do a simpler experiment.

If we send in stuff from the left or send in stuff from the right. And if we solve those problems independently, we can then just superpose the results to get the general solution. Yeah. So it will suffice to always either set d to 0 or a to 0, corresponding to sending things in from the left or sending things in from the right. And then we can just take a general superposition to get the general answer. Everyone happy with that?

So let's do that in this set up. So here's our wave function. And you guys are now adept at solving the energy eigenvalue equation, it's just the matching conditions for a,b,c, and d. I'm not going to solve it for you, I'm just going to write down the results. So the results-- in fact, I'll use a new border. The results for this guy, now for e greater than v, are that-- doo doo doo, what just happened. Right, OK, good. So let's look at the case d equal 0 corresponding to coming in from the left. So in the case of in from the left, c is equal to 2k1 over k1 plus k2, and b is equal to k1 minus k2 over k1 plus k2.

And this tells us, running through the definition of j, the currents, and t and r, gives us that, at the end of the day, the reflection coefficient is equal to k1 minus k2 upon k1 plus k2 [INAUDIBLE] squared, whereas the transmission amplitude or probability is equal to 4k1 k2 over k1 plus k2 squared. And a good exercise for yourself is to re-derive this, you're going to have to do that on the problem set as a warm up for a problem.

So at this point we've got an answer, but it's not terribly satisfying because k1 and k2, what's the-- so let's put this in terms of a more easily interpretable form. k1 and k2 are nothing other than code for the energy and the energy minus the potential. Right, so we must be able to rewrite this purely in terms of the energy in the potential e and v. In fact, if we go through and divide both top and bottom by k1 squared for both r and t, this has a nice expression in terms of the ratio of the potential to the energy. And the nice expression is 1 minus the square root of 1 minus v0 over e over 1 plus the square root of 1 minus v0 over e norm squared.

And suddenly, the transmission amplitude has the form 4 root 1 minus v0 over e over 1 plus the square root of 1 minus v0 over e squared. I don't remember which version of the notes I posted for this year's course. In the version from 2011 there is a typo that had here e over v, and in the version from 2012, that was corrected in the notes to being v0 over e. So I'll check, but let me just warn you about that.

On one or two pages of the notes, at some point, these guys were inverted with respect to each other. But the reason to write this out is now we can plot the following. We can now plot the quantum version of this plot. We can plot the actual quantum transmission as a function of energy, and the classical prediction was that at energy is equal to v0, we should have a step function. But now you can see that this is not a step function, and that's related to the fact that b was not actually 0, as you pointed out.

Hold on one sec. Sorry, was it quick?

AUDIENCE: The notes are wrong.

PROFESSOR: OK, the notes are wrong, good, thank you. That's great. OK, so the notes are wrong. It should be v0 over e, not e over v0 on the notes that are posted. We're in the process of teching them up, so eventually a beautiful set of nice notes will be available. Extra elbow grease. OK.

So what do we actually see? Here we got for e less than v0, we got in d that the transmission probability was exactly 0. So that's a result for the quantum result. But when e is equal to v0, what do we get? Well, when e is equal to v0, this is 1. And we get square root of 1 minus 1, which is square root of 0, over square root of 1 plus square root of 1 minus 1, which is 0. That's 0 over-- OK, good. So that's not so bad.

Except for this 1 over this e, this is a little bit worrying. If you actually plot this guy out, it does this, where it asymptotes to 1. And to see that it asymptotes to 1, to see that it asymptotes to 1, just take e0 gigantic. If you know it's gigantic, this becomes v0 over e, which goes to 0, and if e is much larger than the potential. So this becomes square root of 1. And in the denominator, this becomes square root of 1, 1 plus 1, that's 2 squared, 4, 4 divided by 4, that goes to 1. So for large e much larger than v0, this goes to 1.

So we do recover the classical prediction if we look at energy scales very large compared to the potential height. Yeah? OK. So that's nice. Another thing that's nice to note is that if you take this reflection probability and this transmission probability, then they sum up to 1. This turns out to be a general fact and it's a necessary condition in order for those to be interpretable as reflection in transition probabilities.

The probability that it transmits and the probability that it reflects had better add to 1, or something is eating your particles, which is probably not what you're looking for. So this turns out to be true in this case, just explicitly. But you can also, as you will in your problem set, prove that from that definition r and t, it's always true. A check on the sensibility of our definition.

If it weren't true, it wouldn't tell you that quantum mechanics is wrong, it would tell you that we chose a stupid definition of the transmission and reflection probabilities. So in this case, we actually chose quite an enlightened one. OK, questions at this point? Yeah.

AUDIENCE: Does this analysis even hold for classical particles? If we're talking about the difference between-- like if I threw a baseball and I happened to throw it at a potential that had height one millionth of a joule less than the energy of the baseball, would we observe this also?

PROFESSOR: No.

AUDIENCE: No?

PROFESSOR: No, for the following reason. So here's my classical system. Classical system is literally some hill. So where I was growing up as a kid, there was a hill not far from our house. I'm not even going to go into it, but it was horrible. If you have energy just ever so slightly greater than the hill-- he said from some experience-- if you have energy that's ever so slightly greater-- OK.

I'm going resist the temptation. So one of these days. If you have ever so slightly greater than the hill, and you start up here, what does that tell you? What does it mean to say you have energy just ever so slightly greater than the potential energy at that point?

AUDIENCE: Up more slowly.

PROFESSOR: Little tiny velocity. OK, and I'm going to say I have a little tiny velocity this way. OK. So what happens? We follow Newton's equations. They are totally unambiguous, and what they tell you is, with 100% certainty, this thing will roll and roll and roll, and then all hell will break loose. And then if you're very lucky, your car gets caught in the trees on the side of the cliff, which is later referred to by the policeman who helps you tow it out as nature's guard rail.

I was young, it won't happen again. And So with 100% certainty, Newton's equations send you right off the cliff. And for style points, backward.

So now I'm going to do the second thing. Newton's laws satisfy time reversal invariance. So the time reversal of this is, this thing has this much energy, and it shoots up the cliff. And ever so slowly, it just eventually goes up to the top, where everything is fine. OK, but does it ever reflect back downhill? No. And does it ever-- classically, when you roll the thing-- does it ever actually not go off the cliff and hit the trees, but instead reflect backwards.

I wish the answer were yes. But sadly, the answer is no. Now, if that car had been quantum mechanically small, my insurance would have been much more manageable. But sadly, it wasn't. OK, so this is a pretty stark and vivid-- this is a pretty stark difference between the quantum mechanical prediction and the classical prediction. Everyone cool with that? Other questions? Yeah.

AUDIENCE: I understand why it works out that r plus t equals 1, but I'm not sure I understand the motivation to use the square with the ratios.

PROFESSOR: Excellent, OK. So the reason to do it, so great, excellent. How to say. Here's one way to think about it. The first thing we want is we want a-- so this is a very good question, let me repeat the question. The question is, look, why is it squared? Why isn't it linear? So let's think about that. Ah, ah, ah. The reason it's squared is because of a typo. So let's think about--

[LAUGHTER]

PROFESSOR: Let's step back for second and let's think, OK, it's a very good question. So let's think about what it should be. OK. For the moment, put an arbitrary power there. OK. And let's think about what it should be. Thank you so much for this question. I owe you, like, a plate of cheese or something. That's high praise, guys. France, cheese. So should it be linear or should it be quadratic? What is that supposed to represent? What is T, the capital T, supposed to represent?

AUDIENCE: The probability of transmission.

PROFESSOR: The probability of transmission, exactly. The wave function. Is the wave function, the value of the wave function at some point, is that a probability? It's a--

AUDIENCE: Probability of density.

No, it's the square root.

PROFESSOR: Is it a probability of density? It is a probability amplitude. It is a thing whose norm squared is a probability. So we want something that's quadratic in the wave function, right. Meanwhile, some time before, we wanted a definition of how much stuff is moving past a point in the given moment in time. What's the probability density moving past a point at a given moment in time. And that's where we got the current j from the first place. j was the probability density moving past a point at a given moment in time.

So notice that j is quadratic in the wave function, so it's how much stuff is moving past any particular direction-- we chose it to be the incident to the reflected bit-- at a moment in time. So it's a probability density-- it's actually a current-- and it's quadratic in the wave function. So it has the right units and the right structure to be a probability. If we squared that, we would be in trouble, because it wouldn't be a probability, it would be a probability squared. And in particular, it wouldn't normalize correctly.

So that typo was actually a bad typo. So what happened, now thinking back to-- you have a video someday, so you can scroll back. What happened was I didn't put the squared on the first, I put the squared on the second, and then someone said squared on the first? Like, yes. But the correct answer is no squared on the second. So there are no squareds. Thank you for the question. Very good question.

In particular, the thing that was so awesome about that question was it was motivated by physics. It was like, look, why is this thing squared the right thing? That's two factors of the wave function, it's already quadratic. It should be like a probability. Why are you squaring it? Very good question. Yeah.

AUDIENCE: So in the transmission probability, why do we know it's 0 if the energy is less than v0? I mean, is there any rule?

PROFESSOR: Yeah, OK, good. We'll come back to that question in a little bit, but let me just quickly say that we calculated it, and we see explicitly that it's 0. Now you might say well, look, the true wave function has a little bit of a tail. So there's some [INAUDIBLE] probability you'll find it here if you do a measurement. But the question we want to ask in transmission is, if you send it in from far away over here, how likely are you to catch it far away over here? And the answer is, you are not. That make sense? OK, good.

OK, so so much for that one. Now, here's a fun fact, I'm not going to go through this in any detail. We could have done exactly the same calculation-- OK, we could have done the same calculation sending something in from the right. And sending something in from the right, in on the right, corresponds to d not equal to 0 and nothing coming in from the left, that's a equals 0. So we could have done the case a equals 0, which is in from the right.

And if you do that calculus, what you find is c is equal to k1 minus k2 upon k1 plus k2 d. And b is equal to 2k1 upon k1 plus k2 d. And this says that plugging these guys in, the reflection and the transmission are the same. In particular, physically, what does that mean? That means the reflection and transmission are the same uphill as downhill. Downhill is uphill, they're the same both ways.

But that's truly weird, right? What's the probability to transmit quantum mechanically if you have just a little bit of energy and you send the particle in? How likely are you to go off the cliff. If we have energy ever so slightly greater the potential, and the transmission amplitude is the same downhill as it was uphill, how likely are you to fall off the potential?

AUDIENCE: Always.

PROFESSOR: Never. Because if energy is just slightly greater than v0, then the transmission probability is very, very low. The transmission to go from here to here is extremely low if the energy is close to the height of the barrier. So had I only been on a quantum mechanical hill, I would have been just fine. This is a really striking result, but this thing, the fact that you're unlikely to scatter uphill, that's maybe not so shocking.

But you're really unlikely to scatter downhill, that is surprising. So I'll leave it to you to check that, in fact, the transmission, when we do it in from the right, the transmission and reflection probabilities are the same. Recalling that transmission means going this way, reflection means bouncing back to the right. Cool? Yeah.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Uh, because, again. Jeez, today is just a disaster. Because there's a typo. This should be times a. God. And the reason you know it should be times a, first off, is that these should have the appropriate dimensions. So there should be an appropriate power of a.

And the second thing is that if you double this stuff in, you'd better double this stuff out. So if you double a, you'd better double c. So you know there had better be a factor of a here. And that's just a typo again. I'm sorry, today is a bad day at the chalkboard. Thank you for that question. Yeah.

AUDIENCE: Is there a physical reason for why r and t are the same and so different from what you would classically expect, or is that just the way the math works?

PROFESSOR: There is. I want you to ponder that, and either at the end of today or at the beginning of tomorrow, we'll talk about that more in detail when we've done a little more technology. It's a good question and you should have that tingling sensation in your belly that something is confusing and surprising and requires more explanation, and we'll get there. But I want you to just think about it in the background first. Yeah.

AUDIENCE: Is there any physical experiment where r and t don't sum up to 1?

PROFESSOR: Mm mm. Nope. So the question is, is there any experiment in which r and t don't sum up to 1? And there are two ways to answer that. The first way to answer that is, it turns out that r and t adding up to 1 follows from the definitions. So one of the things you'll do on the problem set is you'll show that using these definitions of the transmission and reflection of probabilities, they necessarily add up to 1, strictly.

They have to. It follows from the Schrodinger equation.

The second answer is, let's think about what it would mean if r and t didn't add up to 1. If r and t didn't add up to 1, then that would say, look, if you throw a particle at a barrier, at some feature in a potential, some wall, then the probability that it goes this way at the end versus the probability that it goes that way at the end if you wait long enough is not 1. So what could have happened? What could have gotten stuck in there?

But that doesn't even conserve momentum, not even approximately. And we know that the expectation of momentum is time independent for a free particle. In between while it's actually in the potential, it's not actually time independent, there's a potential, there's a force.

AUDIENCE: I mean, if it's whole particles, can't some of them get annihilated or something?

PROFESSOR: Ah, well OK. So if you're talking about multiple particles and interactions amongst multiple particles, then it's a slightly more complicated question. The answer there is still yes, it has to be that the total probability in is the total probability out. But we're only going to talk about single particles here. But it's always true that for-- we're going to take this as an assumption, that things don't just disappear, that the number of particles or the total probability is conserved.

There's a third way to say this, this really isn't independent from the first. Remember that these were defined in terms of the current? The current satisfies-- the current concept of probability conservation equation dt of rho is equal to the gradient minus the gradient of the current. So the probability is always conserved, the integral of probability density is always conserved, it's time independent.

AUDIENCE: So it's by [INAUDIBLE].

PROFESSOR: Yeah, it's by construction. Exactly. Yeah?

AUDIENCE: Looking at your expression for the transition probability, I'm having trouble seeing how that works out to 0 when e is less than v0?

PROFESSOR: Oh, this expression is only defined when e is equal to v0. Because we derived this-- excellent question-- we derived this assuming that we had an energy greater than v0 and then that the wave function had this form. You cannot use this form of the wave function if the energy is less than v0. If the energy is less than v0, you've got use that form of the wave function. And in this form of the wave function, we derived that the transition amplitude is 0, because the current on the right, the transmitted current, is 0.

So this calculation is appropriate when e is less than v0 and this calculation is appropriate when e is greater than v0. So you're absolutely correct. You can't use this one e is less than v0, it gives you not the same answer. In fact, it gives you a complex-- it's kind of confusing. The factors cancel. So it's not really a probability at all, and indeed, this is just not the right quantity to use.

AUDIENCE: All right, cool.

PROFESSOR: Cool? All right. So there are a bunch of nice things I want to deduce from what we've done so far. So the first is, look, I pointed out that this can be derived just explicitly and it gives the same results as before. That's not an accident. If you take this system and you just reverse the roles of k1 and k2, what happens? Well that's just replaces e by e minus v0, we can do that by doing this, replacing e by e minus v0 and e minus v0 by e, it swaps the role of a and d, and it gives you exactly the same things back.

So if you're careful about that, you never have to do this calculation. You can just do the appropriate transformation on that calculation and it gives you the exactly the same thing. It's the same algebraic steps.

But the other thing that's nice is that you can actually do the same thing from here. So as long as you set the d equals 0-- so there's another term here that we neglected, the plus delta x, we got rid of it. But you can analytically continue this calculation by noting that look, if we just set alpha, we want minus alpha equals ik2, then the algebra is all going to be the same. We just have ik2 instead of minus alpha. So we can just replace alpha with minus ik2 everywhere in our expressions, being careful about exactly how we do so, being careful to take care of factors of i and such correctly.

And you derive the same results for both cases, which is a nice check on the calculation. So often, when you get a little bit of experience with these, you don't actually have to do the calculation. Again, you can just take what you know from a previous calculation and write down the correct answer. So it's a fun thing to play with, exactly how do you do that. So I invite you to think through that process while you're doing your problem set.

Another thing is the reflection downhill thing which is pretty surprising. But here's the thing that I really want to emphasize. What this calculation shows you is not so much that-- it's not just that transmission downhill is highly unlikely when the energy is very close to the height of the potential barrier. That's true, but it's not the most interesting thing about this calculation.

The most interesting thing about this calculation, to my mind, is the fact that from the detailed shape of the transmission as a function of energy, we can deduce what the potential is. Think about what that tells you. If you do an experiment, you have a barrier, and you want to know the shape of the barrier. Is it straight, is it wiggly, does it have some complicated shape. How do you measure that? Well, you might measure it by just looking. But imagine you can't, for some reason. For whatever, it's in a box or you can't look at it. Maybe it's just preposterously small. How can you deduce what the shape of that hill is?

Well, one way to do it is to send in particles as a function of energy, more and more energy, and measure the probability that they transmit. OK. Now if you do so and you get this graph as a function of e, what do you deduce? You deduce that the barrier that you're scattering off of is a square step with this height v0. Are we cool with that? So apparently, just look at the transmission amplitudes, the transition probabilities, you can deduce at least something of the form of the potential. Which is kind of cool. If you didn't know the potential, you could figure out what it was.

And this turns out to be a very general statement that you can deduce an enormous amount, and as we'll see, you can, in fact, deduce basically everything you want of the potential from knowing about the transmission probabilities as well as the phase shift, the transmission amplitudes.

So this is the basic goal of scattering. And so the way I want you to think about it is imagine, for example, that someone hands you an object. A box. And the box has an in port and it has an out port. And they allow you to send in particles as a function of energy and measure transmission and reflection, you can measure transmission and reflection. Just like I'm measuring transmission off of your faces right now, from the light from above.

So suppose that you do so, put it on your test stand and you measure transmission. You measure transmission as a function of energy, and you observe the following. The transmission as a function of energy is small up until some point, and then at some point, which may be the minimum energy you can meaningfully probe, you get something like this. So here's the transmission as a function of energy. So what can you say?

If this is all the information you have about what's going on inside the box, what can you deduce about the thing inside the box? One thing you can deduce is that it looks kind of like a potential with height around v0. It looks kind of like a potential step with some height v0. This is asymptoting to 1.

However, it's not, because it has these oscillations in it. So there's more to the potential than just a barrier of height v0. What I want to show you is that you can deduce everything about that potential, and that's the point of scattering. So let's do it. So the goal here, again, to say it differently, is what's v0? What is v of x? Not just the height, but the total potential. So another way to say this, let me set up a precise version of this question.

I want to be able to do the following. I want to take a system that has a potential which is constant up to some point which I'll call 0, and then again from some point, which I'll call L, is constant again. And inside, I don't know what the potential is. So in here, there's some unknown potential, v of x, which is some crazy thing. It could be doing anything. It could be some crazy-- it could have horns and whatever. It could be awful. But the potential is constant if you go far enough away, and the potential is constant if you go far enough away.

A good example of this is a hydrogen atom. It's neutral but there's a clearly and complicated potential inside because the proton and the electron are moving around in there in some quantum state, anyway, and if you send something at it, far away, it's as if it's not there. But close by, you know there are strong electrostatic forces. And so the question is what you learn about those forces, what can you learn about the potential by throwing things in from far away, from either side.

Now one thing we know already is that out here, the wave function always-- because it's a constant potential-- always takes the form e to the ikx plus b e to the minus ikx. And out here it takes the form c e to the ikx plus d e to the minus ikx. And again, this corresponds to moving in. d is in from the right. c is out to the right. b is out to the left. And a is in from the left.

So again, there are basically four kind of scattering experiments we can do. We can send things in from the right, which corresponds to setting a to 0. We can send things in from the left, which corresponds to setting t equal to 0. And all the information about what happens in v is going to be encoded in what's coming out, the b and c coefficients.

And the way to make that sharp is just to notice that the transmission probability, if we compute for this system, assuming it's forming the wave function asymptotically away from the potential, the transmission amplitude is just c over a squared when you're sending in from the left. And the reflection is equal to b over a norm squared. This squared is a squared. [INAUDIBLE] function over amplitude squared, good.

To learn about the transmission and reflection coefficients, it's enough-- suffices-- to compute, to know b and c as a function of a and d. All of the scattering information is in those coefficients b and c for a and d. And here I'm assuming that I'm sending in a monochromatic wave with a single, well defined energy. I'm sending in a beam of particles with energy e. I don't know where they are, but I sure know what their momentum is. So some well defined beam of particles with energy e. And these probabilities are going to contain all the data I want.

So this is the basic project of scattering. Questions?

AUDIENCE: So basically, it only depends on the transmission-- it only depends on the edges?

PROFESSOR: That's a good question. The question is, does the transmission depend only on what you do with the edges. And here's the important thing. The transmission depends crucially on what happens in here.

For example, if this is an infinitely high barrier, nothing's going across. So this transmission depends on what's in here. But the point is we can deduce just by looking far away, we can deduce the transmission probability and amplitude just by measuring b and c far away, b and c far away. OK. So the transmission amplitude is something you measure when very far away.

You measure-- if I throw something in from very far out here to the left, how likely is it to get out here very far to the right? And in order to answer that, if someone hands you the answer to that, they must have solved for what's inside. That the point. So knowing the answer to that question encodes information about what happened in between.

AUDIENCE: So I guess initially, your potential's going to-- so say it's stuff you did earlier. The potential drops down to the [INAUDIBLE] in the box. Is that going to be problematic?

PROFESSOR: Yeah, for simplicity, I'm going to assume that the potential always goes to 0 when we're far away, because that's going to be useful for modeling things like hydrogen and, exactly. Carbon, we're going to do diamond later in the semester, that'll be useful. But we could repeat this analysis by adding an extra change in the asymptotic potential, it doesn't really change anything important. Yeah.

AUDIENCE: It looks like even for just the simple step up [INAUDIBLE] you can't tell from just the probabilities why the step is going up or going down.

PROFESSOR: Ah, excellent. Excellent, excellent. So good, thank you for that question, that's really great. So already, it seems like we can't uniquely identify the potential from the transmission probability if the transmission probability is the same for step up or step down. So what's missing?

AUDIENCE: Maybe the energy [INAUDIBLE]

PROFESSOR: But we're working with an energy [INAUDIBLE]. So the energy is just a global constant. We'll see what's missing, and what's missing is something called the phase shift. So very good question, yes.

AUDIENCE: It looks like when we did the example for the step, that t equals [INAUDIBLE] over a squared.

PROFESSOR: Yeah, it is, because it's the ratio. That's why I wrote t is c over here. So t is the ratio of j over ji. And in fact, jt here is equal to c squared times-- so this is the probability density, so it's the probability density times the velocity, what's the effect of velocity here? H bar k 2 over m, whereas j incident is equal to a squared-- probability density times the momentum there, the velocity there-- which is h bar k1 upon m. So here the ratio of j transmitted to j incident is norm c squared k2 over norm a squared k1.

AUDIENCE: So because it's not level--

PROFESSOR: Exactly, it's because it's not level. So here, they happen to be level, so they're only [INAUDIBLE] factor, cancel, and the only thing that survives is the amplitude. Good question. Yeah.

AUDIENCE: For the lead box example, is it sufficient just to know what's reflected back to solve the situation?

PROFESSOR: So for precisely for this reason, it's not sufficient to know t. It's not sufficient to know t and r. But, of course, once you know r, you know t, so you're exactly right. Once you know the reflection probability, you know the transmission probability, but there's one more bit of information which we're going to also need in order to specify the potential, which is going to be the phase shift. But you're right, you don't need to independently compute r and t, you can just compute one.

AUDIENCE: You need sensors on both sides of the box, to answer my question.

PROFESSOR: You don't need sensors on both sides of the box, but you need to do more than just do the counting problem. We'll see that. OK. So let's work out a simple example, the simplest example of a barrier of this kind. We want constant potential, and then ending at 0, and we want a constant again from L going off to infinity. So what's the easiest possible thing we could do? Step, step. We're just doing what we've done before twice. So this is an example of this kind of potential. It's sort of ridiculously simple, but let's work it out.

So we want scattering, let's start out, we could do either scattering from the left or scattering from the right. Let's start out scattering from the left, so d equals 0, and let's study this problem. So what we know-- and let's also note that we have a choice to make. We could either study energy below the height of the potential or we can study energy above the height of the potential. And so for simplicity, I'm also going to start with energy greater than the height of the potential, v0, and then we'll do e less than v0 afterwards. It'll be an easy extension of what we've done.

OK, so this will be our first case to study. So we know the form of the wave function out here, it's a e to the ikx, b to the minus ikx. We know that for the potential out here it's c e to the ikx and d e to the minus ikx. The only thing we don't know is the form of the potential in here. And in here it's actually very simple. It's got to be something of the form-- I think I called it f and g, I did. f e to the ik prime x-- I'm calling this k, so I'll just call this k prime x-- plus g e to the minus ik prime x.

And the reason I chose k prime is because we're working with energy greater than the potential, so this is a classically allowed region. It's an oscillatory domain but with a different k prime. So here, k squared, h bar squared over 2m is equal to e, and h bar squared k prime squared over 2m is equal to e minus v0, which is positive when e is greater than v0. This analysis will not obtain when e is less than v0, we'll have to treat it separately.

So now what do we do? We do the same thing we did before, we just do it twice. We'll do the matching conditions here, the matching conditions here. That's going to give us 1, 2, 3, 4 matching conditions. We have 1, 2, 3, 4, 5, 6 unknown coefficients, so we'll have two independent ones. That's great.

We set d equal to 0 to specify that it's coming in from the left and not from the right, that's 5. And then we have normalization, which is 6, so this should uniquely specify our wave function. Yeah. Once we've fixed e, we have enough conditions.

So I'm not going to go through the derivation because it's just an extension of what we did for the first. It's just a whole bunch of algebra. And let me just emphasize this. The algebra is not interesting. It's just algebra. You have to be able to do it, you have to develop some familiarity with it, and it's easy to get good at this. You just practice. It's just algebra.

But once you get the idea, don't ever do it again. Once you get reasonably quick at it, learn to use Mathematica, Maple, whatever package you want, and use computer algebra to check your analysis. And use your physics to check the answer you get from Mathematica or Maple or whatever you use.

Always check against physical reasonability, but use Mathematica. So posted on the website are Mathematica files that walk through the computation of the transmission and reflection amplitudes and probabilities for this potential, and I think maybe another one, I don't remember exactly. But I encourage you strongly to use computer algebra tools, because it's just a waste of time to spend three hours doing an algebra calculation.

In particular, on your problem set this week, you will do a scattering problem similar to the bound state probably you did last week, the quantum glue problem-- which you may be doing tonight, the one due tomorrow. Which is two delta function wells and find the bound states, so that involves a fair amount of algebra. The scattering problem will involve a similar amount of algebra. Do not do it. Use Mathematica or computer algebra just to simplify your life.

So if we go through and compute the-- so what are we going to do, we're going to use the matching conditions here to determine f and g in terms of a and b, then we'll use the matching conditions here to determine c-- d is 0-- c in terms of f and g. So that's going to give us an effective constraint relating a and b, leaving us with an overall unknown coefficient a, which we'll use for normalization.

The upshot of all of which is the answers are that, I'm not even going to write down-- they're in the notes. Should I write this down? I will skip.

So the upshot is that the transmission amplitude, as a function of k and k prime-- the transmission probability, I should say, is 1 over-- actually, I'm going to need the whole amplitude. Shoot. The transmission probability is equal to-- and this is a horribly long expression-- the transmission probability, which is c over a norm squared, is equal to 4 k squared k prime squared cosine squared of k prime l plus k squared plus k prime squared sine squared of k prime l under 4k squared k prime squared. Seriously.

So we can simplify this out, so you can do some algebra. This is just what you get when you just naively do the algebra. I want to do two things. First off, this is horrible. There's a cosine squared, there's a sine squared, surely we can all be friends and put it together. So let's use some trig. But the second thing, and the more important thing, is I want to put this in dimensionless form. This is horrible. Here we have ks and we have ls, and these all have dimensions, and they're inside the sines and the cosines it's kl. That's good, because this has units of one over length, this has units of length, so that's dimensions. Let's put everything in dimensionless form.

And in particular, what are the parameters of my system? The parameters of my system are, well, there's a mass, there's an h bar, there's a v0, and then there's an energy, and there's a length l. So it's easy to make a dimensionless parameter out of these guys, and a ratio of energies-- a dimensionless ratio of energies-- out of these guys. So I'm going to do that, and the parameters I'm going to use are coming from here.

I'm going to define the parameters g0 squared, which is a dimensionless measure of the depth of the potential. We've actually run into this guy before. 2m l squared v0 over h bar squared. So this is h bar squared, 1 over l squared is k squared over 2m, so that's an energy. So this is a ratio of the height of the potential to the characteristic energy corresponding to length scale l.

So the width, there's an energy corresponding to it because you take a momentum which has 1 over that width. You can build an energy out of that h bar k squared over 2m. And we have an energy which is the height of the barrier, we take the ratio of those. So that's a dimensionless quantity, g0. And the other dimensionless quantity I want to consider is a ratio of the energy, e, to v0, which is what showed up before in our energy plot. Or in our transmission plot over there, e over v0.

So when we take this and we do a little bit of algebra to simplify our life, again, use Mathematica, it's your friend. The result is much more palatable. It's t-- again, for the energy greater than v0-- is equal to, still long. But 1 plus 1 over 4 epsilon, epsilon minus 1 sine squared of g0 square root of epsilon minus 1. And upstairs is a one.

So remember this is only valid for e greater than v0, or equivalently, epsilon greater than 1. And I guess we can put an equal in. So when you get an expression like this, this is as easy as you're going to make this expression. It's not going to get any easier. It's 1 over a sine times a function plus 1. There's really no great way to simplify this. So what you need when you get an expression like this is try to figure out what it's telling you. The useful thing to do is to plot it. So let's just look at this function and see what it's telling us.

Let's plot this t as a function of epsilon and for some fixed g0. Keep in mind that this only makes sense for e greater than v0 or for epsilon greater than 1, so here's 1. And we're going to remain agnostic as to what happens below 1. And just for normalization, we know that the transmission probability can never be greater than 1, so it's got to be between 1 and 0. So here's 0 and 0. We're remaining agnostic about this for the moment.

So let's start thinking about what this plot looks like. First off, what does it look like at 1? So when epsilon goes to 1, this is going to 0, that's bad because it's in a denominator. But upstairs, this is going to 0, and sine squared of something when it's becoming small goes like-- well, sine goes like that thing. So sine squared goes like this quantity squared. So sine squared goes like g0 squared, this goes like 1 over-- this is going like g0 squared epsilon minus 1 square root quantity squared, which is epsilon minus 1.

So this is going to 0 and this, the denominator, is going to 0 exactly in the same way. Epsilon minus 1 from here, epsilon minus 1 downstairs from here. So the epsilon minus ones precisely cancel. From the sine squared we get a g0 squared, and from here we get a 4 epsilon. So we get 1 plus g0 squared over 4 epsilon. But 4 epsilon, what was epsilon here? 1. We're looking at epsilon goes to 1, so this is just g0 squared over 4.

So the height, the value of t-- so we're not looking in here. But at energy is equal to v0 or epsilon is equal to 1, we know that the transmission amplitude is not 0, but it's also strictly smaller than 1. Which is good, because if it were 6, you'd be really worried. So g0 squared, if g0 squared is 0, what do we get? 1. Fantastic, there's no barrier. We just keep right on going through, perfect transmission. If g0 is not zero, however, the transmission is suppressed. Like 1 over g0 squared.

What does this actually look like with some value. And what is this value, it's just 1 over 1 plus 1/4 g0 squared. Cool? OK. And now what happens, for example, for a very large epsilon? Well, when epsilon is gigantic, sine squared of this-- well, sine squared is oscillating rapidly, so that's kind of worrying. If this is very, very large, this is a rapidly oscillating function. However, it's being divided by roughly epsilon squared. So something that goes between 0 and 1 divided by epsilon squared, as epsilon gets large, becomes 0. And so we get 1 over 1 plus 0, we get 1. So for very large values it's asymptoting to 1. So naively, it's going to do something like this.

However, there's this sine squared. And in fact, this is exactly what it would do if we just had the 1 over epsilon times some constant. But in fact, we have the sine squared, and the sine squared is making it wiggle. And the frequency of the wiggle is g0, except that it's not linear in epsilon, it's linear in square root of epsilon. So as epsilon gets larger, the square root of epsilon is getting larger less than linearly. So what that's telling you is if you looked at root epsilon, you would see it with even period. But we don't have root epsilon, we're plugging this as a function of epsilon. So it's not even period, it's getting wider and wider.

Meanwhile, there's a nice fact about this. For special values of epsilon, what happens to sine squared? It goes to zero. And what happens when this is zero? Yeah, t is 1. So every time sine is 0, i.e., for sufficient values when root epsilon is a multiple of pi determined by g0, transmission goes to 1. It becomes perfect. So in fact, instead of doing-- wow. Instead of doing this, what it does is it does this. So let's check that I'm not lying to you. So let me draw it slightly different.

To check, so it's going to 0, and the period of the 0 is getting further and further along. That's because this is square root, not squared. So epsilon has to get much larger to hit the next period. That's why it's getting larger and larger spacing. However, the amplitude goes down from 1, that's when this gets largest. Well, at large values of epsilon, this is suppressed.

As epsilon goes larger and larger, this deviation become smaller and smaller. So that's what this plot is telling us. This plot is telling us a bunch of things. First off, we see that at large energies, we transmit perfectly. That makes sense. This was a finitely high barrier. Large energies, we don't even notice it.

It tells you at low energies-- well, we don't know yet what happens at very low energies. But at reasonably low energies, the transmission is suppressed. And if you sort of squint, this roughly does what we'd expect from the step barrier. However, something really special is happening at special values of epsilon. We're getting perfect transmission.

We get perfect transmission of these points, t is equal to 1. Perfect. Star. Happy with a big nose. I don't know. Perfect transmission at all these points. So this leaves us with a question of why. Why the perfect transmission, what's the mechanism making transmission perfect. But it also does something really lovely for us. Suppose you see a spectrum, you see a transmission amplitude or transmission probability that looks like this. You may get crappy data. You may see that it's smudged out and you see all sorts of messy stuff.

But if you know that there's perfect transmission at some particular epsilon 1, there's more perfect transmission at another epsilon 2, more perfect transmission at epsilon 3, and they scale, they fit to this prediction. What do you know? That you've got a finite high barrier, and it's probably pretty well approximated by a square step. So in your problem set, you're going to get experimental data, and you're going to have to match to the experimental data.

You're going to have to predict something about the potential that created a particular transmission probability distribution. And knowing where these resonances are, where these points of perfect transmission, is going to be very useful for you. Yeah.

AUDIENCE: Wasn't there another potential that will also create periodic 1s?

PROFESSOR: A very good question. So I'm going to turn that around to you. Can you orchestrate a potential that gives you the same thing? That's an interesting empirical question. So, on the problem set, you'll study a double well potential. So the question was, how do I know isn't another potential that gives me the same answer? At this point, we don't know. Maybe there is another potential.

In fact, you can do all sorts of things to make a potential that's very arbitrarily close that gives you an arbitrarily similar profile. So if they're significantly different, how different do they look? So then the students say, well, I don't know. What about a double well potential? So, in fact, we'll be doing a double well potential on the problem set.

A good question. And we certainly haven't proven anything like this is the only potentially that gives you this transmission amplitude. What we can say is if you get transmission amplitude that looks like this, it's probably pretty reasonable to say it's probably well modeled. We seem to be reproducing the data reasonably well. So it's not a proof of anything, but it's a nice model.

And we're physicists, we build models. We don't tell you what's true, we tell you what are good models. If a theorist ever walks up on you and he says, here's the truth, punch him in the gut. That's not how it works. Experimentalists, on the other hand. We'll just punch them too, I guess.

[LAUGHTER]

PROFESSOR: We build models. OK, so this leaves us with an obvious question, which we've answer in the case of energy grade in the potential. What about the case of the energy less than the potential. So we haven't filled out that part of the graph. Let's fill out that part of the graph. What happens if the energy is less than the potential? And for this, I'm going to use the trick that I mentioned earlier that, look, if the energy is less than the potential, that just means that in the intervening regime, in here, instead of being oscillatory, it's going to be exponentially growing [INAUDIBLE] because we're in a classically disallowed region.

So if the energy is less than v0, here we have instead of ikx, we have minus alpha x. And instead of minus ikx we have plus alpha x. So if you go through that whole analysis and plug in those values for the k prime, the answer you get is really quite nice. We find that t for energy less than v0 is equal to-- and it's just a direct analytic continuation of what you get here-- 1 plus 1 over 4 epsilon 1 minus epsilon.

So epsilon is less than 1 now. 1 sinh squared of g0 1 minus epsilon under 1. OK, so now this lets us complete the plot. And you can check, it's pretty straightforward, that because sinh, again, goes like its argument at very small values of its argument, we get g0 squared 1 minus epsilon. Sorry, this should be square root 1 minus epsilon. Really? Is that a typo? Oh no, it's in there. OK, good.

So again, we get g0 squared 1 minus epsilon, the 1 minus epsilon cancels, and we get 1 over 4 g0 squared. Which is good, because if they disagreed, we'd be in real trouble. So they agree, but the sinh squared is just a strictly-- or that function is just a strictly decreasing function as we approach epsilon goes to 0. It goes mostly to 0. So this is what we see. We see an exponential region and then we see oscillations with residences where the period is getting wider and wider.

But this should trouble you a little bit. Here, what are we saying? We're saying look, if we have a barrier, and we send in a particle with energy way below the barrier, that's kind of troubling. If we send in a particle with energy way below the barrier, there's some probability that it gets out. It goes across. So for e less than v0, classically no transmission, we get transmission.

Now, do we get resonances when e is less than v0? No, that's an interesting thing. We'll talk about resonances and where they come from, and we'll talk about the more generalized notion of a nest matrix in the next lecture. But for now I just want to say a couple of things about it. So I want to ask-- so this is called tunneling, this transmission across a disallowed barrier, a classically disallowed barrier-- so this transmission across a disallowed barrier is called tunneling.

And just a quick thing to notice is that if we hold e fixed and we vary l, we vary the width of the barrier, how does the tunneling amplitude depend on the width of the barrier? At fixed energy, if we vary the width of the barrier-- which is only contained in g0-- how does the transmission amplitude vary? And we find that for large l-- for l much greater than 1 for a typical scale in the problem-- the tunneling amplitude goes like e to the minus 2 alpha l where alpha just depends on the energy in the potential.

So what we see is that the probability of transmitting, of tunneling through a wall, depends on the width of that wall. And it depends on it exponentially. The wider the wall, the exponentially less likely you are to tunnel across it. And this fits the simulation we ran where we saw that tunnelling through a very thin wall was actually quite efficient. And we also saw that tunnelling across a wall can have resonant peaks. At special values, it transmits perfectly. We saw that in the simulations.

So at this point I can't resist telling you a very short story. I have a very good friend who is not a physicist, but who is a professor at MIT. So a smart person, and very smart. She's one of the smartest people I know. She broke her ankle. She's always losing stuff, she's constantly losing stuff. And she broke her ankle, and she was going to the rehab place, and she parked her car right in front of the rehab place. She parked her car right in front because she's got the broken ankle and she doesn't want to have to walk a long way to get the car, she parked right in front.

She went in, she did her two hours of rehab, and she came outside, and her car was gone. And she's always misplacing her car, she's always misplacing her keys, she's always losing everything. But this time, this time she knew where it was. And it wasn't there. So first she thought, oh crap, my car's been stolen. How annoying, I've got a broken ankle. But she looks around, and there's her car on the other side of the street pointed the opposite direction.

And she's like, finally. Finally. She was explaining this to me and a friend of mine who is also a physicist, she said finally, I knew I had incontrovertible proof that quantum effects happen to macroscopic objects. My car tunnelled across the street.

[LAUGHTER]

PROFESSOR: At this point, of course, my friend and I are just dying of laughter. But she said my car tunnelled across the street. But here's the problem, I couldn't tell anyone. Because if I told anyone, they would know that I was crazy. Clearly I'm crazy, because I lose my stuff all the time, but surely it's just gotten misplaced.

So I went home, and I got home and I was sitting down to dinner with my partner and our daughter and I was burning up inside because I wanted to tell them this crazy thing happened. I know quantum mechanics, it happens. I couldn't tell them anything, so I was just fidgeting and dying. And finally my daughter said, mom, didn't you notice that we moved the car across the street?

[LAUGHTER]

PROFESSOR: See you next time.