8.04: Quantum Mechanics
Massachusetts Institute of Technology

Professor Allan Adams
Wednesday April 10, 2013

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# Problem Set 8

*Due Wednesday April 17, 2013 at 10.00AM*

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**Assigned Reading**:

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|---|---|
| E&R | $7_{all}$, $\text{App}_{M,N}$ |
| Li. | $8_6$, $9_{1,2,3}$, $10_{1,2,3}$ |
| Ga. | $7_{all}$, $8_{1,4,5}$ |
| Sh. | $12_{all}$ |

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1. **(15 points) Superposition State of a Free Particle in 3D**

   At time $t = 0$, a free particle in 3d ($V(x, y, z) = 0$) is in the superposition state,

   $$\psi(\vec{r}, 0) = \frac{\pi^{-3/2}}{2\,L^{3/2}} \sin(3x/L)\, e^{i(5y+z)/L} \ .$$

   (a) If the energy of the particle is measured at $t = 0$, what value is found?

   (b) What possible values of the momentum $\vec{p} = (p_x, p_y, p_z)$ will measurement find at $t = 0$ and with what probabilities will these values occur?

   (c) Given the state $\psi(\vec{r}, 0)$ above, what is $\psi(\vec{r}, t)$?

   (d) If $\vec{p}$ is measured to be $\vec{p} = \frac{\hbar}{L}\,(3\hat{e_x} + 5\hat{e_y} + \hat{e_z})$at $t = 0$, what is $\psi(\vec{r}, t)$?

2. **(15 points) Degeneracies**

*Suppose a system has some symmetry – e.g. rotational symmetry. This means the energy doesn't change upon acting with the symmetry – e.g. if you rotate the system. This generally implies that the set of energy eigenvalues is* degenerate, *i.e. that there are multiple independent eigenstates sharing the same energy. The* degeneracy *of the system at some energy refers to the number of energy eigenstates which share that energy eigenvalue.*

(a) Consider a free particle in 1d with definite energy $E = \frac{\hbar^2 k^2}{2m}$.
  i. How many linearly independent states share this energy?
  ii. What symmetry guarantees this degeneracy?

(b) Consider a 2d harmonic oscillator with frequencies $\omega_x = \omega_y = \omega$.
  i. What are the energy eigenvalues?
  ii. What is the degeneracy of the $n^{th}$ eigenvalue?
  iii. What symmetry guarantees this degeneracy?

(c) Now suppose we nudge the system so that $\omega_x = (1 + \epsilon)\omega$, with $\epsilon \ll 1$.
  i. What are the new energy eigenvalues?
  ii. Is the spectrum again degenerate?
  iii. Plot the first 6 eigenenergies as a function of $\epsilon$ for $-0.1 \leq \epsilon \leq 0.1$.
  iv. What is the relationship between the breaking of symmetry and the splitting of degeneracies?

3. **(20 points) Mathematical Preliminaries: Angular Momentum Operators**

In classical mechanics, the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ is conserved in any rotationally symmetric system. In QM, the angular momenta are given by operators:

$$\hat{L}_x = \hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y \;, \qquad \hat{L}_y = \hat{z}\,\hat{p}_x - \hat{x}\,\hat{p}_z \;, \qquad \hat{L}_z = \hat{x}\,\hat{p}_y - \hat{y}\,\hat{p}_x \;.$$

(a) Using the basic commutator relations, $[\hat{x}_a, \hat{p}_b] = i\hbar\,\delta_{ab}$, show that[1]:

$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x \;, \qquad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y \;, \qquad [\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z \;.$$

(b) Consider the operator $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ . Using your results above, show that[2]:

$$[\hat{L}_z, \hat{L}^2] = 0 \;.$$

Argue, without further calculation, that $[\hat{L}_x, \hat{L}^2]$ and $[\hat{L}_y, \hat{L}^2]$ must also vanish.

(c) Consider the "ladder" operators $\hat{L}_- = \hat{L}_x - i\hat{L}_y$ and $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$. Use the above commutation relations to show that:

$$[\hat{L}_z, \hat{L}_\pm] = \pm\hbar\,\hat{L}_\pm \;, \qquad [\hat{L}^2, \hat{L}_\pm] = 0 \;.$$

What properties of the eigenvalues of $\hat{L}^2$ and $\hat{L}_z$ follow from these commutators?

[1] This matters because *operators which do not commute do not have the same eigenvectors*! More precisely, given operators $\hat{A}$ and $\hat{B}$ and a state $\psi$, it follows that $\Delta A \Delta B \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle_\psi|$. Since eigenstates correspond to definite eigenvalues with zero uncertainty, a state $\psi$ can only be a simultaneous eigenstate of $\hat{A}$ and of $\hat{B}$ if $\langle[\hat{A}, \hat{B}]\rangle_\psi = 0$. In general, then, the commutator $[\hat{L}_x, \hat{L}_y] = i\hbar\,\hat{L}_z$ tells us that having a definite value of $\hat{L}_x$ means you generally *don't* have a definite value of $\hat{L}_y$ – just like having a definite position means *not* having a definite momentum. See Shankar, Chapter 9, for a beautiful disquisition.

[2] This matters because it tells us we *can* find states with definite values of both $\hat{L}^2$ and $\hat{L}_z$. Meanwhile, as we discussed in lecture and as you can check by using $\vec{\hat{p}} = -i\hbar\vec{\nabla}$ and working in spherical coordinates,

$$\hat{L}^2 = -\hbar^2 \left[ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right] \;.$$

But that's just the operator we ran into in lecture inside the energy operator for a central potential! Indeed, in spherical coordinates, $\hat{p}^2 = \frac{1}{r}\partial_r^2 r + \frac{1}{r^2}\hat{L}^2$. For central potentials, eigenstates of $\hat{E}$ are eigenstates of $\hat{L}^2$.

4. **(15 points) Mathematical Preliminaries: Eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$**

   *In the previous problem we exploited the commutation relations amongst the coordinates and their momenta to determine the commutation relations amongst the components of the angular momentum. In this problem we will use the explicit coordinate representations of the angular momenta ($\hat{\vec{L}} = \vec{r} \times \vec{p} = \frac{\hbar}{i}\vec{r} \times \vec{\ }$) in spherical coordinates to study the eigenfunctions, $Y_{lm}(\theta, \phi)$, of the angular momentum operators.*

   Working in spherical coordinates, $\hat{L}^2$ and $\hat{L}_z$ take the form,

$$\hat{L}^2 = -\hbar^2 \quad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta} \quad \sin\theta\frac{\partial}{\partial\theta} \quad + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \quad , \qquad \hat{L}_z = -i\hbar\frac{\partial}{\partial\phi} \, .$$

   and the first few spherical harmonics, $Y_{lm}$, take the form,

$$Y_{0,0} = \sqrt{\frac{1}{4\pi}} \, , \qquad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta \, , \qquad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \, .$$

   (a) Show that these functions are properly normalized and orthogonal to one another.

   (b) Show that these functions are eigenfunctions of both $\hat{L}^2$ and $\hat{L}_z$, and compute the corresponding eigenvalues.

   (c) Construct $Y_{42,-41}$.

5. **(15 points) Angular Momenta and Uncertainty**

   The commutator relations you derived in Problem 3 above imply an important set of uncertainty relations amongst the angular momenta,

$$\Delta L_x \Delta L_y \geq \frac{\hbar}{2}|\langle \hat{L_z} \rangle|\,, \qquad \Delta L_y \Delta L_z \geq \frac{\hbar}{2}|\langle \hat{L_x} \rangle|\,, \qquad \Delta L_z \Delta L_x \geq \frac{\hbar}{2}|\langle \hat{L_y} \rangle|\,.$$

   Consider a particle in a normalized eigenstate of $\hat{L}^2$ and $\hat{L}_z$, $\Psi \propto Y_{lm}$ and $(\Psi|\Psi)$=1.

   (a) Show that in this case $\langle L_x \rangle = \langle L_y \rangle = 0$. *Hint: use the operators* $\hat{L}_+$ *and* $\hat{L}_-$.

   (b) Show that $\langle L_y^2 \rangle = \langle L_x^2 \rangle = \frac{\hbar^2}{2}\left[l(l+1) - m^2\right]$. *Hint: use* $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$.

   (c) Using your results, verify the first uncertainty relation above. Can the uncertainty in any two components of $\vec{L}$ ever vanish simultaneously?

6. **(20 points) Lifting the Degeneracy of the Quantum Rigid Rotor**

Consider a spherically symmetric rigid rotor with moment of inertia $I_x = I_y = I_z = I$. For example, it might help to imagine Professor Evans curled up into a compact and uniform sphere[3] and set spinning. Classically, his energy is given by,

$$E = \frac{\vec{L}^2}{2I}$$

(a) What are the energy eigenstates and eigenvalues for this quantum rigid rotor?

(b) What is the degeneracy of the $n^{th}$ energy eigenvalue?

Now suppose Prof. Evans gets sore and stretches a bit such that his moment of inertia in the $z$ direction becomes $I_z = (1 + \epsilon)I$, with the other two moments unchanged.

(c) What are the new energy eigenstates and eigenvalues?

(d) Sketch the spectrum of energy eigenvalues as a function of $\epsilon$. For what sign of $\epsilon$ do the energy eigenvalues get closer together? Intuitively, why?

(e) What is the degeneracy of the $n^{th}$ energy eigenvalue? Is the degeneracy fully lifted? If so, explain why and suggest a way to break only some of the degeneracy. If not, explain why not and suggest a way to break all of the degeneracy.

[3] A Yoga master, he is not, but strong with the centripetal force, he is.

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