

The following content is provided under a Creative Commons license. Your support will help MIT OpenCourseWare continue to offer high-quality educational resources for free. To make a donation or to view additional materials from hundreds of MIT courses, visit MIT OpenCourseWare at ocw.mit.edu.

PROFESSOR: All right, so for the next six lectures, including today, we're going to finish off the course with the application of everything we've studied so far to a couple of ideas. The first being the existence of solids-- why we have solids and why we have conductivity in solids, which is basic properties of materials. In particular, the story I want you guys to leave the course with is an understanding of why diamonds are transparent and why copper isn't, which is sort of a crude fact about the world. But we can explain it from first principles, which is pretty awesome.

The second is we're going to come back to this idea of spin and of $1/2$ integral angular momentum. This intrinsic angular momentum of the electron. And we're going to use that to motivate a couple of ideas. First off, we're going to tie back to Bell's inequality from the very beginning, and then we're going to do a little bit on quantum computation. That will be the last two lectures.

So before I get started though with today, two things. First, I'm going to ask for questions. But the second is I got some really good questions about hydrogen, so I want to wrap up with one last comment on hydrogen because it's entertaining. But before I get started on that, questions about everything up till now? Yeah.

AUDIENCE: So in the [INAUDIBLE] there's something called spin orbit couplings, another edition, the [INAUDIBLE], that we haven't talked about, in addition throughout this progression. Is that something we can do with our knowledge now? Or is something [INAUDIBLE]?

PROFESSOR: It absolutely is. So the question is-- and this is a very good question-- the question is, look, if you look on Wikipedia about-- seriously, this is a rational thing to do. If you look at Wikipedia to learn something about hydrogen, what you discover is

everything we've talked about, including the fine structure, including the Zeeman effect. But you see that there are a couple of other effects.

So one effect, for example, is when you look at the Zeeman effect a little more carefully than we did, there's a second term in the Zeeman effect, which is an induced magnetic moment. So even when the angular momentum is smaller or even zero, there's still a contribution to the energy from the externally imposed magnetic field. So there are lots of other corrections.

One of them in particular is something called spin orbit coupling. And so the question was, what is that? And do we know enough to explain what spin orbit coupling is?

So let me give you a very short answer to that question. So we know that if we write down the energy operator for our system, we have the energy operator-- so the full energy operator for hydrogen, I'll say, is the energy operator for coulomb plus a bunch of corrections. So for example, we had the relativistic correction plus a term that was a coefficient times p to the fourth. And this is on your problem set. And this came from relativistic corrections. I shouldn't call this c . I'll call it beta, maybe. Some coefficient.

And there are a whole bunch of other contributions. For example, there's the term we studied that contributed to the Zeeman effect. If there's an external magnetic field, there's a $\mathbf{B} \cdot$ magnetic moment of the system. And if the electron is in a state with finite angular momentum, the magnetic moment is some dimensionful coefficient, which we usually call the Bohr magneton just because-- whatever-- we liked the guy. Times the angular momentum.

But there can be additional contributions. There are a whole bunch of additional terms in the true potential for hydrogen. One of them takes the following form. It's a constant plus-- maybe I'll call it κ times spin dotted into the angular momentum of the electron, of the electron bound to the hydrogen system.

So where this comes from is a sort of beautiful story. So first off, could it be there?

Sure. This is a term that could exist with some coefficient. Why not?

It turns out that you can derive its existence from a study of the relativistic version of the hydrogen system. We're not going to study that in any detail in 804, but the basic idea and what this does is totally amenable to 804-level analysis. This is just saying that when you have some orbital momentum, or when you have some spin angular momentum, both of those corresponds to angular momentum of an object that carries charge. And so it's not unreasonable that there's going to be some interaction between the two where the magnetic moments-- those two magnetic moments either want to be aligned or anti-aligned depending on the sign of this coefficient.

However, we haven't talked about spin in any great detail yet. We're going to do that in the last week of the course. So I'm going to defer talking about this in any detail. And we're only going to discuss it briefly at the end for a couple of weeks. But absolutely, these additional couplings are important for hydrogen and they're also things you can study at this level, at 804 level.

I should also emphasize that there are a whole bunch of other corrections. There are lots of terms. One set of terms, an infinite number of terms, are the further sub-leading relativistic corrections of kinetic energy. And there are a whole bunch of other corrections, so I'm going to leave those out for the moment. But I want to emphasize to everyone that we're building models of hydrogen, and they're all approximations. Yeah.

AUDIENCE: So it's a question regarding this [INAUDIBLE].

PROFESSOR: Yes. There is. So on the problem set, you're asked to estimate the correction to the energy. I wrote down the answer in lecture the other day. You're asked to compute, in particular, the l dependence. But to estimate the magnitude or to estimate the value of the shift in the energy eigenvalues to hydrogen from the first relativistic correction is p to the fourth correction.

And I expect that this will be covered in your recitation in some detail, but there the

question was, is there a trick to make this a little easier?

There are a couple of good physical arguments, which I'm not going to tell you. But there are nice ways to do this. But at the end of the day, you do not want to end up doing the following computation.

You don't want to take the expectation value of p to the fourth and write this as the sum integral of ψ complex conjugate derivative to the fourth ψ . If you attempt to do this integral, you will weep. This is not the way you want to do it.

And here's what you want to do. And I'm not going to tell you how to get here, but you want to reduce the calculation of this expectation value to the calculation of the expectation value of $1/r$ and of $1/r^2$. And it turns out that if you know these expectation values, that entirely suffices to compute the expectation value of p to the fourth if you take into account the fact that the energy eigenfunctions satisfy the original energy eigenvalue equation. So p^2 upon $2m$ plus v of x of v of r is equal to e when acting on those wave functions.

So for computing these guys, you could try to do this brute force, not unrelated to problem, I believe, two on Rydberg atoms on the problem set. But there are actually sneakier ways to do these expectation values as well. And I will leave that to you.

But let me emphasize that you can do a direct brute force calculation, but you don't need to. And I would encourage you to try to find an efficient, indirect way to do these calculations. Did that answer your question? OK. Anything else? Yeah.

AUDIENCE: So is [INAUDIBLE] a time dependence to an operator with the [INAUDIBLE] being [INAUDIBLE]?

PROFESSOR: Does adding-- well, it depends on how you introduce the time dependents.

AUDIENCE: Can a operator [INAUDIBLE]?

PROFESSOR: Yes. Yeah, absolutely. What's the reason for the question?

AUDIENCE: One of the problems here.

PROFESSOR: One of the problems in the problem set?

AUDIENCE: Yeah. It's like the last part of four.

PROFESSOR: Remind me which problem this is, I don't remember.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Oh. Oh, oh! Ha, ha, ha. Sorry, I really like that problem. Let me rephrase your question in the following way. Is it intrinsically non-Hermitian to have time dependence in a system?

So no, you can have-- so what does Hermitian mean? Physically, what does it mean for the energy to be Hermitian for example?

What it means for the energy operator to be Hermitian is that time evolution, which is represented by the oper-- so let me phrase it this way. So if we know that the energy operator is Hermitian, what is that telling us?

Well, it tells us a lot of things. It tells us the energy eigenvalues are real. That's good. So E^\dagger is equal to E . But it tells us something else.

Remember that the Schrodinger equation is that $i\hbar \frac{d}{dt} \psi$ is equal to $E \psi$. And we use this to argue that the general solution to the Schrodinger equation can be written in the following elegant way, $\psi(t)$ is equal to $e^{-iHt/\hbar} \psi(0)$, where this was the evolution operator $U(t)$. Yeah?

And this is a unitary operator. And the way they say its unitary goes back to the exam. The way to say it's unitary is this is $e^{-iHt/\hbar}$ is a real number, H is a real number. So this is $e^{-iHt/\hbar}$ is Hermitian operator. And anything of the form $e^{-iHt/\hbar}$ is unitary. It's adjoint to its inverse.

What that tells you is that since this is unitary, it preserves the magnitude, or the norm, of the wave function. So probability is conserved. What would it mean for the energy operator to be not Hermitian?

Well, it would mean a lot of things. One thing it would mean is that the energy eigenvalues are no longer real. That's a little weird. But the much more troubling thing is that the energy would no longer be-- sorry, the probability would no longer be conserved. The evolution operator, which is the solution to the Schrodinger equation, would no longer be a unitary operator. And the probability, the norm or the wave function, would no longer be conserved.

So the question at the end of-- the last question in problem four is really asking, in the system you're thinking about is, probability conserved? And that's the question that you should be asking yourself when you finish up problem four. Good? Yeah.

AUDIENCE: When we solve the Schrodinger equation in that way, does this solution come from the fact that E is Hermitian or that E is like that nice little [INAUDIBLE], that E is time dependent. And that if E was time dependent, why couldn't the coefficients for the probability also be?

PROFESSOR: Absolutely. So here I was just focusing on the Hermiticity. And in solving this, I'm assuming that the energy is time independent. If the energy is not time independent, then this is not the right answer as you're pointing out. So in fact, you have to do an integral and you have to time order things, and it's a complicated story. But this is not the solution.

But even if we have a time independent system, if the energy operator is not Hermitian, this is not unitary. So indeed, you're absolutely right that if the energy operator is time dependent, the story's more complicated than just this. But we already see the problem that's salient for problem four at this stage with a time independent operator. Yeah.

AUDIENCE: Just one question please. So if I had some sort of system which is [INAUDIBLE], some of my particles are weak in the other process.

PROFESSOR: Yeah.

AUDIENCE: And now the probability is not conserved. Saying that this thing is not [INAUDIBLE], the issue is, I think, which is also extremely disturbing for [INAUDIBLE] complex in

energy, [INAUDIBLE], what is that?

PROFESSOR: Yeah.

AUDIENCE: That seems even worse.

PROFESSOR: Well, I guess it's a matter of taste. I would say that they're the same thing in the following sense. Suppose I have complex energy eigenvalues.

Well, I know that if I have a stationary state, then that stationary state as a function of time is equal to the stationary state at time 0 times $e^{-iEt/\hbar}$ so this is an energy eigenfunction rather than-- is $e^{-iEt/\hbar}$ the energy times t over \hbar ? Or this is the energy eigenvalue.

Now, imagine E is complex. Sorry, first imagine E is real. That's not hard to imagine because it's usually the case. And if that's true, what happens to the wave function as we evolve through time?

It rotates by phase. Now, if E is complex, imagine E is of the form $E - i\gamma$ let E be complex. So E is going to be E real minus $i\gamma$, some imaginary piece. So what that's going to give me is an $e^{-iEt/\hbar}$. And that's a decaying thing that when you get to norm squared gives you a decaying envelope over time. It's going to give you an exponential decay. This is going to be equal to $e^{-iEt/\hbar}$ times $e^{-\gamma t/\hbar}$. And when we take the norm squared, the phase goes away, but this doesn't. So having loss of probability is having a complex energy eigenvalue. Yeah.

AUDIENCE: Why are we talking about time independent operators if there's an electromagnetic field and stuff is clearly going on between the avenue of the electromagnetic field?

PROFESSOR: This is a very good question. And that actually leads me into the thing I wanted to talk about first. So I got a bunch of questions-- thanks for that question. I got a bunch of questions over the past few days about the magnetic moment of the hydrogen system and what a strange idea that is. So let me talk about that for a second. And when I'm done with this little spiel, tell me if I've answered your

question.

Well, let's just leave this up. OK. So in particular, let's think about this term for a moment. Where this term came from last time was we said, look, we turn on an external magnetic field. Someone turns on the switch and current runs through an electromagnet and we get a uniform magnetic field in our fiducial volume, where we're doing the experiment.

And the electron system, if it carries some angular momentum, it also has a charge, angular momentum charge. That means it's got a magnetic moment. That's how much magnetic moment. So this is saying that the magnetic moment of the electron bound to the proton with angular momentum wants to anti-align with the magnetic field. Or align in this case because I put the wrong sign.

But here's something really upsetting about that. So what I just said sounds crazy if you think about it in the following way. Take an electron in the coulomb system, just straight up coulomb. Take an electron in the coulomb system, put it in, say, the ground state, lowest energy state. Is it moving? This is problem four. So is it moving?

The expectation value of position doesn't change in time. It's a stationary state. Expectation value of nothing changes in time. OK, fine, but it's the ground state that carries your angular momentum. That's not so upsetting. Go to the first excited state with angular momentum. The n equals 2, l equals 1, m equals 1 state. So it's got as much angular momentum in l_z as possible. Is that thing moving?

Yeah. It's not moving at all. And yet, we're saying there's a current associated with it, an electric current. And that electric current is inducing a magnetic moment, a la right-hand rule. Maybe via Biot-Savart, if you want to be fancy. And that magnetic moment-- a little current loop-- is leading to this interaction, the Zeeman's interaction.

We know that it's true because experiments show the Zeeman splitting. So it is definitely true that there is a magnetic moment of this thing. But it's not moving, how

can that be? So how can there be a current if nothing is moving?

AUDIENCE: It is, in some sense moving, [INAUDIBLE]. Why is the expectation value of the [INAUDIBLE] to stop moving?

PROFESSOR: What's the expectation value of the momentum?

AUDIENCE: I think it's also probability.

PROFESSOR: Yeah, it's 0.

AUDIENCE: I mean, first of all, the world isn't classical. You can't use classical intuition.

PROFESSOR: OK, true.

AUDIENCE: Second of all, let's suppose you have a uniform ring of classical charge and set it spinning.

PROFESSOR: Yes.

AUDIENCE: That ring is as a ring, not moving, but there's still a current associated with it.

PROFESSOR: So you're saying that the electron is a ring?

AUDIENCE: It might make sense if you give it that [INAUDIBLE].

PROFESSOR: I like this. No, this is good. It's wrong, but it's good. And the reason it's good is that you're really pushing your assumptions to try to figure out how the experimental data can possibly match. And you're saying, look, we have to just reject our intuition. Our intuition is clearly leading us astray. And I like that. That's correct. So what we're going to do in the next few minutes is work through that and try to find the best way to phrase that. Your strategy is the correct one. So let me rephrase that slightly.

Look, if you have a classical distribution of charge, that distribution of charge could be a stationary distribution-- a distribution, which as a distribution of charge doesn't change a dime. But each individual charge in that distribution is itself moving. The

problem here is that we just have the one electron.

If you ever look, you will find the electron at a spot. But what you're really saying is, look, it's a mistake to think about the electron having a definite position in the first place. You just shouldn't think about that. The best you can say is that it has some probability of being in any spot. So let's work with that. Let's take that idea and let's push it. Let's see how far we can take this idea that the electron has some probability [INAUDIBLE].

So suppose we have an electron in a stationary state of the coulomb potential. The stationary states are labeled by three integers-- n , l , and m . And we'd written their wave functions. There are, of course, an infinite number of ways to write in different notation. There are an infinite number of fonts one could use for the normalization constant $1/r R_{nl}$ of little r , Y_{lm} of theta and phi.

And I'm going to rewrite this. I'm going to expand this out slightly. This is $N 1/r R_{nl}$ of R and l . And Y_{lm} , remember, was of the form P_l^m of cosine theta times $e^{im\phi}$. Everyone cool with that?

So I just wrote the spherical harmonic as a polynomial in cosine theta times an exponential in phi.

AUDIENCE: Where did the $1/r$ come from, on the left?

PROFESSOR: The $1/r$ was-- so most people call capital R the whole thing. But this is the pulling out to $1/r$ to simplify the radio wave equation. And the reason I prefer writing it this way is just that this guy satisfies a simple 1d Schrodinger equation.

AUDIENCE: That was the u of r ?

PROFESSOR: That was a thing that we called u of r , and then I confused the hell out of everyone by calling three different things on the board u , which was sort of unnecessary. So this is the artist previously known as u . The dude can sing.

OK, so the first question I want to ask is, so this is a stationary state. Is the electron moving?

No, not in any conventional sense. If you compute the expectation value of the position and you take its time derivative, this is zero. We could do this either by calculating it or just by observing that it's a stationary state. And on principle, it can't change in time. So this guy is not moving in any conventional sense.

So why is there an electric current? Why do we get the Zeeman magnetic moment, the Bohr magneton?

So as was pointed out, look, this is quantum mechanics. It's not classical mechanics. And in quantum mechanics, the electron isn't at any point. Rather, there's some probability density. And the probability density that we find the electron at some point r is ψ^2 or ψ^* ψ . A familiar beast.

And meanwhile, a wonderful thing about the probability distribution in quantum mechanics that we've already discussed here is that it's conserved. The time derivative of the probability density or ρ -- remember, this is a density, not a probability. The time rate of change of the probability distribution is minus the gradient-- the divergence of the probability current, where the probability current j is equal to \hbar over the mass, which I'm going to call the mass m . Oh, gee, I don't want-- I'm going to call the mass capital M . That's the mass of the electron, which is a little strange because it's a very small quantity.

But anyway, \hbar over capital M of the imaginary part of ψ^* gradient ψ . And you showed on a problem set long ago that this is true if j takes this form by virtue of the Schrodinger equation.

Now, we usually write this imaginary $1/2i$ times ψ^* gradient-- or ψ^* gradient ψ minus ψ gradient ψ^* , but that's equal to the imaginary part of the first term. The thing you want to emphasize is the imaginary part.

So we have this current. In our system, the position expectation value is time independent. And indeed, it's a stationary state. So beyond the position expectation value being time independent, the probability density itself is time independent

because the wave function evolves by an overall phase and the probability density is the norm squared. So the phase goes away.

So in our system, in an electron in this state in the coulomb potential, the time rate of change of the probability density-- let me actually do this over here. So in the stationary state, $\psi_{l, n, m}$, the time rate of change of the density is 0. And this tells us by the conservation equation that the divergence of the current is also 0. Does that tell us that the current is 0?

AUDIENCE: No.

PROFESSOR: Right. So what's the current?

Well, we've written everything here in spherical coordinates. And the current is given in terms of the gradient operator. So let me remind you quickly of what the gradient operator is in spherical coordinates. It has three components-- a radial component, a theta component, and then a phi component. And the radial part is just D_r . The theta component is $1/r D_\theta$. And the phi around the equator component is equal to $1/r \sin \theta D_\phi$. Everyone cool with that? So what's j ?

Well, j is going to have a component, j_r in the radial direction. The current in the radial direction. And intuitively, what should that be? Is there stuff going out or in?

There shouldn't be. It's hydrogen. It's not doing this. So this should be 0. And we can just quickly see that this is \hbar upon M imaginary part of-- well, this is the r component, it's going to be the derivative in the radial direction. But the derivative in the radial direction is going to be real. The derivative in the radial direction is going to be real.

So when we take the norm squared, we don't pick up anything. We pick up an overall coefficient. It's going to be strictly real. The phase, $e^{i m \phi}$ cancels out because this is the complex conjugate. So the imaginary part of this thing is going to be 0. OK? So J_r is 0.

And similarly, J_θ is a real function. When we take a derivative, we get a real

function.

And then when we multiply by its complex conjugate, again, the phase cancels out and we just get a whole bunch of real stuff whose imaginary piece is 0. So J_θ is 0. But J_ϕ is a cool one.

In my head, I was just thinking J_ϕ . OK, at least someone got that.

So J_ϕ , however, is not going to be 0 for the following reason. So what is it equal to?

It's equal to \hbar upon M -- the mass, M -- times the imaginary part of-- well, ψ^* , which is ψ complex conjugate. And then the gradient with respect to ϕ . The ϕ component is $1/r \sin \theta$. And then derivative with respect to ϕ . The derivative with respect to ϕ pulls down a $i m$.

Oh, look-- times ψ , which is equal to \hbar upon m , times-- well, ψ is norm squared. That's real. ψ^2 . $1/r \sin \theta$. That's real, so I can pull that out. And we are left with imaginary part of im , which is just m , which is not 0. And in particular, it's proportional to $\hbar m$.

Everyone see that? So what this is saying is that the current, the probability current, for an electron in the stationary state ψ_{nlm} of the Coulomb potential is equal to $\text{norm } \psi^2 \text{ over } m r \sin \theta \hbar m$ in the ϕ direction. The ϕ unit-vector. Cool?

So nothing is moving, but there's a current. What is moving? What is the thing of whose current-- of whom this is the current?

AUDIENCE: Probability density.

PROFESSOR: The probability density. The probability density is rotating. So what this is telling us is that it's true the system is stationary. But I want to know-- I'm looking down at the equatorial plane, OK.

So this plot is going to be looking down on the equatorial plane of hydrogen. Here is

the origin, the center of the potential. And what this is telling-- so I'm going to draw these vectors. They're in $e\phi$ direction. OK. But their magnitude falls off with $1/r$ and the norm square of the wave function, which also is falling off exponentially.

And it goes to 0 at the origin as r to the $l+1$. And m must be no bigger than l . So in order for m to be non-zero, l must be non-zero. Which means this must go like r to the something greater than $m+1$. So the r 's cancel out. And we have that it vanishes. So the contribution vanishes at the origin, is small near the origin, is largest at some radius, and then falls off again.

So the magnitude of the arrow here is meant to indicate the magnitude of the flux. So what we see is that we have a probability distribution where the probability, as a distribution, is time independent. Right? The probability distribution is, in fact, time independent. But there's a current which is a persistent, stationary current. Everyone cool with that?

So if I ask you, where is the particle? Well, the probability density tells you that. And if I ask you, what's the current? The electric current, minus e times j . At a point. And this is really the current density and the probability current density.

Now what is the consequence of the statement, the probability density itself is not changing in time. It's that there is zero divergence of this current. And indeed, there is zero divergence. Every little bit going into a spot is matched by some equivalent amount going out. There's zero divergence. Cool?

And so we see that while it's true that nothing is moving, it is also true that there is a non-trivial electric current. And when you use the Biot-Savart law to sum up the contributions from each little bit of this current, what you get is-- right hand rule-- a magnetic field, a magnetic moment. And following nothing but the Biot-Savart law and using what you know about the wave functions, this gives us that the magnetic moment is equal to the Bohr magneton, \hbar times m_z . OK? Everyone cool with that? Yeah?

AUDIENCE: So when the perimeter intuitively is [INAUDIBLE], still make all these other

expectation values?

PROFESSOR: Yeah.

AUDIENCE: Why is that an intuitive force? Or maybe it's not.

PROFESSOR: Well, here's one way to say it. There are two-- let me rephrase that question. Let me give you a different question to ask. And let's think about the answers to that question and I hope that will answer your question, if it doesn't ask it again. So here's the question I would suggest that you ask. Or that someone ask. I'll ask it. And I just totally lost my train of thought. That's totally me. What was the question I wanted you to ask? Wow, that's amazing. I completely just in the blink of an eye totally lost my train of -- ah, yes.

So here, it was crucial in this calculation that the current was given by the imaginary part of the gradient. And as you showed in a problem set a while back, anytime you have a wave function which is real, then the current will vanish. Here it was crucial-- in order to get a non-vanishing current-- it was crucial that the wave function was not real. It had a phase.

But you also showed on a problem set that you can always take your energy eigenfunctions-- for bounce-- you can always take your energy eigenfunctions and make them real. You can always construct a basis of energy eigenfunctions which are strictly real. Sorry?

AUDIENCE: In one dimension, right?

PROFESSOR: Yeah well--

AUDIENCE: Aren't you in three also?

PROFESSOR: What did you use for that proof?

AUDIENCE: Just used linear combination of the two.

PROFESSOR: Yeah. You just use the energy eigenvalue equation, hermiticity, and the linear, the

energy operator. And that's perfectly true in any number of dimensions.

AUDIENCE: OK.

PROFESSOR: So that sounds crazy. So first off, why do we have a not real energy eigenfunction? Can we construct purely real energy eigenfunctions? If we can, those energy eigenfunctions would appear to have no current. So they would have no magnetic moment. How do these fit together? Can you construct energy eigenfunctions that are pure real for this system, for the Coulomb potential?

Let me give you a hint. In 1d for a free particle, what are the energy eigenfunctions?
1d,

AUDIENCE: [INTERPOSING VOICES].

PROFESSOR: e^{ikx} , right? That's not real. But there's another eigenfunction, you get the minus ikx . And if you take the sum of those, you get e^{ikx} plus e^{-ikx} divide by 2 for fun-- and that gives you cosine of kx . That's real. But there's a second one, which is sine of kx , which is also real. Now you can take linear combinations of them with i 's and get the exponentials back. But let's just take the real part, right?

Now, do those carry any momentum? What's the momentum expectation value for cosine of kx ?

AUDIENCE: 0.

PROFESSOR: 0, because you get confirmation from plus k and one from minus k . They exactly cancel. Now look at this solution. Can you construct a real energy eigenfunction of the Coulomb potential?

AUDIENCE: Sure.

PROFESSOR: How.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Great.. Let's take this and its complex conjugate. So what that would be? Well, if we take this, ψ_{nlm} . And its complex conjugate, well, what is its complex conjugate? These are real. That just gives me a minus. So that's the state plus ψ_{nl} and i minus m , which is also an allowed state with the same energy by rotational invariance. And this is thus the state ψ_{nl} . And it doesn't have a definite L_z angular momentum, right? So I'll just call it $\psi_{sub\ nl}$ unhappy face.

It is not an L_z eigenfunction. But it is an energy eigenfunction. And it could also have built a minus with divide by $2i$ for fun. I could have built the sine or the cosine of ϕ . One of which showed up on your exam. So I can find a basis of these states, sine and cosine, instead of, give me the $i\psi_{nl}$, I need the minus $i\psi_{nl}$. And those would have been perfectly real. And in those situations, what would the current have been?

AUDIENCE: [INAUDIBLE].

PROFESSOR: If I put the system in this stationary state, what would the current have been?

AUDIENCE: 0.

PROFESSOR: Identically 0. However, that wouldn't be a state with a definite angular momentum. So it wouldn't be surprising that it has zero current. It's got some contribution. It's a super position of having some angular momentum and having the opposite angular momentum. And the expectation value, the expected value, is zero.

I can also study states with a definite value of the angular momentum. Those are not real. There's nothing wrong with that. I could have constructed states which don't have a definite angular momentum but are real. But instead, I want to work with states that have a definite angular momentum. That cool? So there's no tension. There's no contradiction.

It's just that, if we're interested in finding states that have a definite energy and a definite L_z angular momentum, that is not going to be a real function. And nothing tells us that it has to be. And when you have a non-trivial L_z , when you have a non-trivial angular momentum, just like a classical particle with nontrivial angular

momentum that has charge, we find that there's a current. And thus a magnetic moment.

So the important thing with having a definite nonzero momentum, or in this case angular momentum, to give us the current. Yeah? Questions about this? Yeah?

AUDIENCE: Using the same logic, it kind of looks like we're taking it a charged particle and moving it around in circles. Like, where we have a distribution and we're spinning.

PROFESSOR: Yeah.

AUDIENCE: With spin a distribution of charge, we're accelerating the charged particles in it.

PROFESSOR: Yes.

AUDIENCE: Individually.

PROFESSOR: Yes.

AUDIENCE: If we accelerate them, they emit radiation.

PROFESSOR: Yes.

AUDIENCE: Hydrogen atoms don't emit radiation.

PROFESSOR: Yes.

AUDIENCE: How does that go?

PROFESSOR: That sounds a lot like problem four on your problem set.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Ah, OK. Well, that's a great question. And you should think about it. How could I say-- you read my mind. Yes. Struggling with this is exactly the point of one of the problems on your problem set. And ask me that again after the problem set has been turned in, and I'll give you a happy disquisition on it. But I want you to struggle with it. Because it's hard and interesting question. Yeah?

AUDIENCE: So I don't think this answers my question necessarily about the radiation. Because, analogously, the classical em, this would be the magnetostatic case, where you just have a static--

PROFESSOR: Precisely. Precisely.

AUDIENCE: But with radiation, you have, quickly oscillating fields and the transition times are really small and the frequencies of the wave functions are really fast. So I still don't know if that's the same.

PROFESSOR: So this question is unfortunately a linear combination of a really good independent question and that question. So ask me after the lecture and I'll talk to you about the part that's linearly independent. OK. I'm on thin ice here. OK. Anything else before we dispense with hydrogen? OK.

The question that you all keep coming to about the-- well, problem four on the-- I think it's problem four on the problem set. Maybe it's problem two. It involves almost no computation, but it's the most intellectually challenging problem on the exam-- on the problem set. So take it seriously, even though it's calculation-free. OK. Yeah?

AUDIENCE: Due to this proton in the [INAUDIBLE].

PROFESSOR: Yeah. Yeah. It's so that the-- it's not by symmetry, because there isn't a symmetry. The proton is 2,000 times heavier than the electron. But-- let me make sure I'm understanding your question. Your question is, we found that the electron is in bound states in the Coulomb potential. But in hydrogen, you have two parts, you have an electron and a proton. So is the electron in that state and the proton is just free and cruising on its own thing? That's exactly the topic of the next 45 minutes. OK. Good question.

So with that insight, let me turn now to the question of identical particles or multiple particles. OK. So we're done with Coulomb for the moment. Pretty much for the rest of the course. So I want to move on to the following question. Suppose I have a system-- we've spent a lot of time thinking about a particle in a potential.

I would like to think about multiple particles for a minute. And neat things happen for multiple particles that don't happen for individual, isolated particles. So let's think about what those-- let's think about the physics of multiple particles.

So in particular, classically-- in classical mechanics-- if I have two particles, what is the information I have to specify? I have to specify the state. I have to specify the position of the first particle. x_1 and its momentum, p_1 . And then the position of the second particle x_2 and p_2 . I'm going to omit vectors over everything, but everything is in the appropriate number of dimensions.

So in classical mechanics-- in quantum mechanics, the state of the system is specified by a wave function, which is a function of the positions-- or the degrees of freedom, let's say x_1 and x_2 . And x_1 and x_2 and p_1 and p_2 are promoted to operators representing these observables.

And the wave function is a function-- now let me just quickly tell you what this notation means. What this notation means is x_1 is some number, like 7. And the quantity in the first spot, this means that the first particle indicated by the first slot, is at x_1 . And the second particle is at x_2 .

So whenever I write a wave function, what I mean is this is the probability amplitude-- the thing whose norm squared is probability-- to find the first particle at this value and the second particle of that value. So the 1 and 2 label the points, not the particles. The particle is labeled by the position inside this wave function. Everyone cool with that? OK. So that's just a notation.

So the quantum mechanical description of two particles is a wave function of both positions and operators representing the position and the coordinates of the particles. And the probability-- actually, let's go ahead and finish up here-- and the probability density to find the first particle. First at x_1 , and the second at x_2 . It's just the norm squared of the amplitude, $\psi(x_1, x_2)$, norm squared. Just as usual. That's the probability density. All right? Everyone cool with that?

And what are x_1 , x_2 , commutator? What is this commutator?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Can you know the position of the first particle and the position of the second particle? Simultaneously? Sure. I'm here. You're there. 0. And x_1 with p_1 -- just work in one dimension for the moment-- is equal to?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Our \hbar . And the same would have been true of x_2 and p_2 . And what about x_1 with p_2 ? 0. These are independent quantities. OK. So it's more or less as you'd expect.

So just to be explicit about this, let's do an example. Two free particles. So what's the energy operator? Well, the energy operator is equal to p^2 upon $2m$ for the first particle, 1, plus p^2 upon $2m_2$. Second particle, its mass is m_2 . The first particle has mass m_1 . And the system is free, so this is plus 0. Yeah?

AUDIENCE: So for the [INAUDIBLE] exponentially, is that true [INAUDIBLE] particles [INAUDIBLE]?

PROFESSOR: Yeah. OK. So the question is, x_1, x_2 , commutator equals 0? Is that true even if there are forces between the particles? So what are the forces between particles-- what are they going to contribute to? Yeah, the potential. And that's going to show up in the energy operator. So that will certainly matter and it will change what the energy eigenvalues are. But it won't tell what can or can't be measured. It won't tell you what properties the system can have or not. Good. Yeah?

AUDIENCE: What's with the particles in the state in which measuring the position of one makes the position of the other one in certainty?

PROFESSOR: We'll talk about things like that in a minute.

AUDIENCE: Does it screw this up?

PROFESSOR: No. This is-- do the commutation relations care what state you're in? They're

relations amongst operators. So they're independent of the state always. OK. Yeah?

AUDIENCE: For the two free particles, are we assuming that they're not charged particles?

PROFESSOR: Yeah. I'm assuming that there are no interactions whatsoever. Just totally free particles. Potential is zero. V of x_1, x_2 equals 0. So they're not charged. They're just totally uninteresting. No interactions, no forces, no potential.

OK. So we know how to solve this problem because we can use separation. Ψ of x_1, x_2 can be written as-- well, we know what the solutions of p_1 upon $2m$ is equal to e^{ikx} . So we can write this as χ of x_1 . And actually, I'm going to call the positions of these guys a and b . Because the subscriptions are just terribly misleading.

So the first point is called a , the second point is called b . So a and b . And this is p_1 and p_2 the momentum of the two guys. χ of a and ϕ of b .

So all I'm doing here is I'm using separation. p_a, p_b . So here's, instead of calling the positions x_1 and x_2 , I'm going to call them a and b just because it's going to be easier to write and make things clear. So I'm using separation. And then p_1^2 squared, or p_a^2 squared, upon $2m$ on χ of a is equal to $e^{ik_a a}$, χ of a . And ditto for b .

This tells us that χ of a is equal to $e^{ik_a a}$, $e^{ik_a a}$.

Everyone cool with that? A is just replacing x_1 . So it's just the position. So this is just saying that the solution to a single free particle is a plane wave. And because it's just the sum of the momentum squared, we can separate the equation and the same thing obtains for ψ . So similarly for ϕ . Ditto. ϕ of b is equal to the $e^{ik_b b}$. I'll call this k_b to distinguish them.

So the wave function's a basis for the wave functions for two particles, ψ of a and b with energy E is equal to $e^{ik_a a} A + e^{ik_b b} B$, Where E is equal to $\hbar^2 (k_a^2 + k_b^2)$ upon $2m$. Yeah?

For a system of two free particles, is every wave function of the form χ of a times ϕ of b ? If you have a system that's separable, is every wave function, is every solution itself, separated?

AUDIENCE: No.

PROFESSOR: No. Because we can have arbitrary superpositions of forms of this type. So we get superpositions of plane waves as long as the energies of each plane wave in the superposition are equal to E -- the total energy is equal to E -- I can just superpose them and I still have an energy eigenfunction. Everyone cool with that? OK. So nothing shocking here. But I did this example so that we'd have the notation of χ and ϕ for the two states.

OK. So now let me come to this question of what about the proton? Well suppose I have a system now which is not a free particle. It's two particles, a and b . And E is equal to $p_a^2 / 2m_a + p_b^2 / 2m_b$. Plus a potential that depends only on $a - b$. OK?

So this is, for example, what happens in the Coulomb potential when you include the proton having a finite mass instead of being infinitely massive and stuck still. So now we can do exactly the same thing we did in classical mechanics when you have a potential that only depends on the distance between two things. I can reorganize degrees of freedom into the center of mass position.

R is equal to $1 / (m_a + m_b)$ of $m_a a + m_b b$. So that's the center of mass motion. And the relative distance is equal to $1 / (m_a - m_b)$ whoops. I don't need that. $A - b$. And then if you do this and you write out the energy operator, E is equal to $-\hbar^2 / (2M) d^2 / dr^2 + V(r)$. OK?

So this is exactly what happens in classical mechanics. You work in terms of the center of mass coordinate and the relative coordinate. The relative coordinate becomes effectively an independent degree of freedom with a potential, which is the central potential. And the center of mass coordinate is a free particle. So if we have a proton and our electron and they're attracted to each other by Coulomb there's a center of mass motion. And then they do together whatever they do together, a la Coulomb potential. Everyone cool with that?

The only difference is that the mass in the Coulomb potential is not the mass of the bare electron. But it's the geometric mean μ is equal to m_a and b over m_a plus m_b . Now for a proton and an electron, if m_a is the proton and m_b is the electron, then this is proton electron over proton plus electron. But proton is about 2,000 times the mass of the electron. So this is basically the mass of the proton and they factor out. So the effective reduced mass is roughly equal to the electron mass. Corrections of a part in 1,000. OK?

So to answer your question, is the proton also in some complicated state described it? Well in fact, neither the electron nor the proton are described by the Coulomb potential. But the relative position, the relative radial distance between them, is controlled by the Coulomb potential and the center of mass degree of freedom is a free particle. Does that make sense? OK. Good.

OK. So, so much for that example. The free particle and the central potential. Here's the much more interesting that happens when we have multiple particles. Yeah?

AUDIENCE: Can you explain what you mean by the [INAUDIBLE]?

PROFESSOR: Oh, yeah, sorry. This is the-- I generated the gradient with respect to r . So if r is a vector, this is the gradient with respect to r , norm squared. And this is the gradient with respect to the relative coordinate. So for example, this is-- if we're in one dimension, so this is strictly one dimensional. Then this is just the derivative with respect to r squared. And this is derivative with respect to little r squared. Just the gradient squared. Did that answer your question?

AUDIENCE: Yeah. What about in hydrogen?

PROFESSOR: Well then it's gradient operator. The thing that takes the function gives you a vector which is the directional derivative of that--

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah. Exactly. In the r direction.

AUDIENCE: Is that subtraction right there?

PROFESSOR: Sorry?

AUDIENCE: Is that a subtraction between--

PROFESSOR: A subtraction. Where?

AUDIENCE: Between the first and second [INAUDIBLE].

PROFESSOR: Minus \hbar squared upon $2m$, dr squared. Minus \hbar squared over 2μ , dr -- this?

AUDIENCE: OK, so that's-- those are separate terms, not multiplied?

PROFESSOR: Yeah. They're not multiplied, right? It's just the sum. So the energy is kinetic energy of the relative-- of the center of mass. Kinetic energy of the relative degree of freedom. And then potential energy. Which is exactly what happens in classical mechanics. Yeah?

AUDIENCE: What's that circle under the first term?

PROFESSOR: This one? M ? It's the total mass. m_a plus m_b . And this one is the reduced mass $m_a m_b$ upon m_a [INAUDIBLE].

OK. So here's a cool thing that happens with multiple particles that didn't happen previously. Suppose we have identical particles. So in particular, imagine I have two billiard balls. So I have two billiard balls and I shoot-- I send one in from one side. And I send in the other in from the other side. And then they collide and there's some horrible chaos that happens. And one goes flying out to this position, a . And the other goes flying out to this position, b . OK?

Now here's my question. Which ball went to a and which ball went to b ? Well if we did this experiment, that would be easy to answer. Because we could paint a little 1 on this one and a little 2 on this one and they'd go flying out. And then at the end, when you catch the ball at a and you catch the ball at b , you can grab them and

look at them and say aha, this one in my left hand which I got from a has 1 on it, and this one has 2 on it, and I'm done.

The other way we could have observed which ball went where is we could've taken a high speed film of this and watched frame by frame and said aha, particle 1, particle 1, particle 1, particle 1. Particle 2, 2, 2, 2. Right? We could have just followed the paths. And we haven't done anything to the experiment. We just took a film. We haven't messed with it. We don't change the results of the experiment. We just watch. Right? Perfectly doable classically.

Quantum mechanically, is this doable?

AUDIENCE: No.

PROFESSOR: No. Because first off, if you watch carefully along and figure out did it go through this slit, did it go through that slit? You know you change the results. 100% white versus 50-50. If you go back to the boxes.

And meanwhile, if they're truly identical particles like electrons, there's no way to paint anything on the damn particle. They're just electrons. And they're completely, as far as anyone's ever been able to tell, completely and utterly identical. They cannot be distinguished in any way whatsoever. So you can't do the thing where you grab the one from a and grab the one from b and say aha, this one had the 1 on it. They're indistinguishable. Yeah.

AUDIENCE: [INAUDIBLE]

PROFESSOR: We're going to come to the Pauli exclusion principle. Hold on to that. Hold on to that question. We're going to come-- we're going to get there. OK.

So we can't-- if we have truly identical particles, what we mean by that is there's no way to run this experiment and determine which particle ended up-- which of these two particles went to a and which went to b. Everyone cool with that? They're identical. And what I want to understand is, what are the consequences?

So the first and basic consequence of this is that the probability that the first particle

ends up at a and the second particle ends up at b must be equal to the probability that the first particle ends up at b and the second particle ends up at a. Because you can't tell which is which. If you can't tell, it must be that those probabilities are equal. Because if they weren't equal, you effectively have skewed the results and they're distinguishable.

These are totally indistinguishable. So the probability that the first particle ends up at a, second at b, must be equal to the probability the first particle ends up at b and the second ends up at a, because you cannot tell the difference. This is what it means to be identical. That equals sign is what I mean by saying I have identical particles. Cool?

So let's find out what the consequences of this are. Define the following operator. And Dave Larson, if you're watching, this is the [INAUDIBLE] [INAUDIBLE] p . So I'm going to call this operator script-y $p_{1,2}$. It's the operator-- so it's got a little hat on it, one more offense-- which takes the first particle and the second particle and swaps them. OK?

So-- and I guess I don't even need the 1, 2. This p operator swaps particle 1 and particle 2. So for example, it takes probability of a to b to probability of b to a. But more importantly, this swapping operation takes the wave function, the amplitude, of a and b, and it swaps a and b, ψ of ba.

Now it's clear that the probability-- that the swapping operation does nothing to the probability, because the fact that they're identical particles means that these are equal to each other. So it hasn't changed the answer. But just because the probabilities are equal to each other, does that tell you that the wave function is invariant under swapping the particles?

AUDIENCE: No.

PROFESSOR: No. It doesn't have to be invariant. The important thing is that the norm squared of ψ . So in principle, this could be equal to some phase, $e^{i\theta}$ of ψ_{ab} . Let me call it θ_{ab} . ψ of a,b. And if this was the case, that there was a phase

that we got here, when we take the norm squared, the probability remains the same. OK?

On the other hand, we know something else. We know that if we take the wave function ψ_{ab} . And we swap it and then we swap it again, what do we get? I have ψ_{ab} . But that means this is equal to $e^{2i\theta} \psi_{ab}$. So swapping twice had better give me 1, so this had better be equal to 1. Let me write this slightly differently. This is $e^{i\theta}$ squared is equal to 1.

So what must be true of $e^{i\theta}$ of our phase? It's a number that squares to 1. So it could be one of two values. It could be 1 or it could be minus 1. That's it. So what that tells us is-- ψ_{ab} . So that tells us that P on ψ_{ab} is equal plus or minus ψ_{ba} . Sorry, ψ_{ab} .

Another way to say this is just that-- another way to say this is that P squared acting on ψ is just ψ again. So the eigenvalues of P have to be plus or minus 1. Here they are. So in fact, this is-- let me phrase this in a little more correct way. This tells us that the eigenvalues of P are plus and minus 1. Yeah?

AUDIENCE: I'm trying to figure out how it can be minus 1? If P squared is ψ then it [INAUDIBLE] has to be ψ [INAUDIBLE]?

PROFESSOR: Yes.

AUDIENCE: How could we add that other [P ? ?]

PROFESSOR: Good. OK. So let's check this quickly. So the question is how can it be minus 1. How can that-- that doesn't-- that would seem to violate our calculations. So what this is saying is that, we know if we take P on, let's say, ψ along ψ of ab . Let's say-- let ψ be an eigenfunction of P . OK? So if ψ is an eigenfunction of P , with eigenvalue minus 1, then this P on ψ is equal to minus ψ . Yeah?

Now the probability is equal to ψ squared. And so P on ψ -- sorry, P on ψ squared is equal to minus ψ squared. We take each side, it goes to minus ψ . So this is just equal to ψ squared. Yeah? Who asked the question? Sorry. Right. OK.

So it leaves the norm squared invariant. So it's OK to have a minus 1 eigenvalue under p , because that doesn't change the probability distribution. The probability distribution is left invariant. However, if we take p squared on ψ , that's equal to p on p on ψ , is equal to the p on minus ψ , which is equal to minus minus ψ , which is equal to ψ . OK? So this is the statement that the square acts as the identity. Did that answer your question?

AUDIENCE: [INAUDIBLE].

PROFESSOR: OK. OK.

AUDIENCE: Along the [INAUDIBLE], we need p squared on side B to go back to side B. It would preserve the probability regardless if we had had the two [INAUDIBLE].

PROFESSOR: Good. Why did p squared have to be the identity? Because what is p doing? p takes two particles and it swaps them.

AUDIENCE: Oh.

PROFESSOR: And if it swaps them again, what do you get? The original configuration. Right? If by swapping, if by p , you mean the thing that swaps those particles, then doing it twice is like not doing anything at all. You can define a different quantity which isn't this swapping operation which does this twice and gives you something else. That's perfectly reasonable. But I'm going to be interested in the operator, which is just swap. And if you do it twice, you get back to the identity.

AUDIENCE: Oh. OK.

PROFESSOR: Cool? OK. Yeah?

AUDIENCE: So regarding the defaults, is what the operation is doing is changed particle number 1, particle number 2, [INAUDIBLE]?

PROFESSOR: Well--

AUDIENCE: What does it do with [INAUDIBLE]?

PROFESSOR: Right. So at any-- given any configuration, at any moment time, right, pick your wave function, pick your state. For example, two particles here. What the script p operator with swapping operator does, it just swaps them. So it swaps the position of one and the position of the other.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Right. Exactly. And that-- you do that, at some moment in time. You do that to a state. So in that experiment, what-- I mean, there's no answer the question, you know, does p swap them before or does it swap them after. It's up to you. You can apply the p operator anytime you want. Cool? OK. Yeah?

AUDIENCE: So when you extracted the probabilities, obviously if you have a case where it's really far apart and the two particles end up [INAUDIBLE], aren't they more likely that they have not moved the entire distance in between?

PROFESSOR: Yeah, that sounds reasonable.

AUDIENCE: It sounds like--

PROFESSOR: Yeah. This is disconcerting.

AUDIENCE: --kind of like the wave function. [INAUDIBLE].

PROFESSOR: That's exactly right. So that's an excellent observation. Let me rephrase that slightly. So here's the observation that she's making. It's exactly correct and it's where we're going to get in a few minutes. So the observation is this.

Look, imagine I take two electrons, which for the moment we'll just call identical particles, so we take two identical particles. Put one in my right hand, one in my left hand. And I just hold them there and I wait for a while. A while later, is it the same electron in my right hand?

AUDIENCE: Probably.

PROFESSOR: I don't know. They're identical. I can't tell. They're completely identical. And so if you

think about this like you do some like weak scattering-- if you did some weak scattering process between these where you pick them very far apart, very slowly moving along-- but they're very far away so the electrostatic interaction is very small and so they repel each other just a little bit.

If this system is, in fact, identical and if the wave function is, let's say for the moment-- and we'll talk about whether this is correct or not-- if the wave function is invariant under swapping the particles-- let's just imagine that it's invariant and they're swapping the particles-- then there are two things that could have happened. The particles could have done this. Or there's also a contribution where they do this. Which kind of hurts.

And in order for the system to be symmetric, you have to have both contributions. So let's come to that. But indeed, it's as if there's some additional interactions, or some additional correlations. And that's exactly what we want to study. So let's get to that. Very good observation.

So what I'd like to do is make that precise. So there are two kinds of particles-- or three kinds of particles, I should say-- in the world from this point of view. The first kind of particle are distinguishable particles. Suppose I have two particles, one with a mass m and one with a mass $2000m$. Say just to pick randomly a number. Right? Those are distinguishable because you can weigh them. So you can tell which one is the heavy one, which one is the light one. And you can tell. Cool?

So there are distinguishable particles. And if we have distinguishable particles-- I'll call $\psi_{\text{sub } d}$ for distinguishable-- then it's OK to have the following thing. $\psi_{\text{sub } d}$ distinguishable first particle is-- the amplitude for the first particle would be an a and the amplitude for the second particle would be a b , could be χ of a , some function of a , and ϕ of b .

This is not invariant under a goes to b . Because under a goes to b , it becomes χ of b , some function of b , ϕ of a , some different function of a . That's distinct. But it doesn't matter. They're distinguishable. So that's perfectly fine. It's not true p of ab is not equal to p of ba . But that's OK, because they're distinguishable. Everyone

cool with that?

AUDIENCE: So p of ba is i of p ?

PROFESSOR: Yeah. Exactly. So p of ab , this is by definition equal to norm squared of ψ of d -- and I should say d . Norm squared of ψ of d of ab squared, which is equal to norm squared of χ of a , ϕ of b squared. Whereas this guy would have been χ of b , ϕ of a , norm squared. And since those are just some stupid functions-- I haven't told you what they are, just some random functions-- then they're just different probability distributions.

So on the other hand, if we have indistinguishable wave functions, then ψ indistinguishable, we know that ψ squared of a,b squared is equal to ψ of ba norm squared. And this is not of that form. So we have two possibilities. I'll write this as ψ plus minus.

If I know one of the particles is in the state described by χ , and the other particle is in this state described by ϕ , this would be an example of a wave function with that property. However, it's not invariant under swapping a and b . So how could I make it invariant under swapping a and b ? Well I could do the following. $\frac{1}{\sqrt{2}}$, χ of a , ϕ of b . And if I want to make it invariant, I could add plus χ of b , ϕ of a .

And now if I swap a and b , here this becomes χ of b , ϕ of a , but that's exactly this term. And this becomes χ of a , ϕ of b . That's this term. So just swap them. Yeah? But we don't need the wave function to be invariant under swapping. We just need it to be invariant under-- up to a sign.

So the other option is to have a minus sign here. And this gives us that the swapping operation, p , acting on ψ plus minus of a,b is equal to plus minus ψ of a,b . And the plus is generally called the symmetric, and the minus, the anti-symmetric combination. OK.

So distinguishable particles can just be in some random state, but there are constraints on what states, what combinations of states are allowed, for

indistinguishable particles. If you can't tell the difference between two indistinguishable particles and you know one is in the state χ and the other in the state ϕ , this cannot be the wave function. It must be either $\chi\phi$ plus $\chi\phi$ in this fashion, or minus. Everyone cool with that? Yeah?

AUDIENCE: I have a question about what we mean by p of ab . So normally, when we talk about probabilities we say that, yes, if you measure a system what's the probability that the metric value will equal that?

PROFESSOR: Yes.

AUDIENCE: But if we can't even determine anything from that type of system, what do we have a probability of?

PROFESSOR: Good. So what this probability means is, what's the probability that if-- that upon observation in the system, I find the first particle to be at a and the second part to be at b . Right? And I can check that by saying like, look, I catch the first particle. I catch the second particle. Is this a ? No. OK. Then that gives zero to the probability distribution. I do that a billion times and I build up statistics.

And if I'm a fourth-- one out of four times, I'll find a particle, the first particle-- or I'll find a particle at a . One out of four times, I'll find a partial at b . And the probability that I find the first particle at a and the second particle at b is one tenth, say. OK.

AUDIENCE: So we can never make that if they're identical, right?

PROFESSOR: What you can't do if they're identical is you can't say which particle you caught at a . This is saying a particle at a or a particle at b , right? But it's-- but whether this is the same or not of probability that I find a particle at b -- the first particle a b and the second particle at a . If you can't tell the difference then they're just a particle at a and a particle at b . OK. OK.

So what does this give us? So this gives us a couple of nice facts. So imagine-- that's an exciting sound. So this gives us a couple of nice facts with which we can find awesomeness in the world.

The first is the following. If you have identical particles, then the energy can't depend on the order. If you have identical particles and they're truly identical, you swap them, then the energy will be the same. If it wasn't the same, then they're distinguishable by figuring out what the energy is. So in order that they're identical, it must be true that if you swap the particles, and then compute the energy, this should be the same as what you get if you first compute the energy and then swap them.

Which is to say that the commutator of e with the swapping operator, p , is 0. OK? But what that tells you is that the expectation value of p doesn't change in time. In particular, if it's some initial state-- if you were initially p on ψ is equal to plus ψ at time 0, then ψ -- then p at ψ , $p \psi$, is equal to plus ψ for all future times. $P \psi$ of p is equal to plus ψ of t . OK.

So if you have two identical particles and the wave function is invariant under swapping them at some moment in time, then it will always be invariant under swapping them. It's a persistent property of particles that the wave function is invariant under swapping them. Yeah? Yeah.

AUDIENCE: What about things that like, become indistinguishable. For example, you have atomic nuclei like, for uranium. And one of them is in like a heavier isotope. And during the time that you're like, holding them in your hands one of them decays, now they're the same isotope.

PROFESSOR: Yeah. That's-- OK.

AUDIENCE: [INAUDIBLE].

PROFESSOR: This tells you something. This is a very good question. So the question is, suppose I have an excited isotope of uranium that's distinguishable from some other isotope of uranium. I wait for a while and then this thing decays down to the state-- it won't decay to the stabilized isotope of uranium, but whatever-- it decays down to-- we could imagine a universe in which it did. It decays down to a stable state. And-- I mean, uranium's never stable. But anyway, you get the idea.

At which point they're indistinguishable. This sounds better. Because now the wave function should be invariant. But it started out not being invariant. What's the problem in this argument? The system has changed. In particular, something went flying out. So this is actually kind of a nice way to argue that there must have been something else. The wave function describes a full system. But if something leaves, then it's not the same system anymore and the wave functions isn't describing the same degrees of freedom. Something left.

AUDIENCE: But suppose we keep track of that one, we still can't swap the two uraniums.

PROFESSOR: Exactly. Then-- well, then there's some additional constraint, right? It must be invariant under swapping that. But it must-- but the wave function also knows about that extra bit that went flying off. And so the whole wave function has to be invariant under swapping the identical parts, but not invariant under swap-- the invariance is not just those two things. They're correlated with that thing that went flying away.

So another way to think about this is imagine two different hydrogen atoms. Here I've got a hydrogen atom. It's an electron and a proton. Here's a deuterium atom. It's an electron bound to a proton and a neutron glued together, deuteron. So are the electrons identical? Yeah, they're totally identical.

So is the wave function invariant-- does the wave function-- or the probability distribution have to be invariant under swapping the electrons? Yes, they're identical. So the probability distribution must be invariant under swapping the electrons. However, is an identical under swapping the hydrogen with the deuterium?

AUDIENCE: No.

PROFESSOR: No. So it's invariant under swapping the identical parts and not the non-identical parts. Cool?

AUDIENCE: Yeah.

PROFESSOR: OK. So this tells us that there are two kinds of particles. There are persistent-- or

sorry, there are three kinds of particles. And these properties are persistent. The first kind of particle-- sets of particles-- are distinguishable particles.

If you have two particles which are distinguishable then you're done. They're distinguishable. Nothing else to say. Two, you can have identical particles with the property that if you take p on $\psi_{a,b}$ -- sorry, p on ψ . If you swap the particles, this is equal to plus ψ . And then you have-- so identical with plus.

And you have three identical particles where if you swap the particles, you get a minus sign on ψ . These particles are called-- we have a name for particles of this kind. Bosons. These are called fermions.

My TA did a bad thing to me when I was taking quantum mechanics. And said, just imagine them as little tiny Fermis. So just take a picture of Fermi in your head and imagine little tiny-- and this is cruel, because I can't help it. Every time someone in a seminar is like, blah, blah, blah, Fermi. And I'm like, damn it. Little Fermi. It's really quite annoying. So now you have it too. Great.

So what are the consequences of the fact that there are two kinds of particles in the universe? These fermions and bosons? This has a really lovely consequence. The first is, suppose we have two fermions. OK? Examples of fermions are electrons. Suppose we have a wave function for two fermions.

The first might-- what's that probability amplitude that the first is at a and the second is at b ? Well it's this ψ of a,b . And the statement that it's a fermion is the statement that this is equal to minus ψ of b, a . If we swap the positions of the two particles, we must pick up a minus sign. This tells us in particular that the probability amplitude for the first particle to be at a and the second particle to be at a is equal to minus itself. Because upon swapping the particles, we get minus ψ of a,a . So the probability amplitude to find two fermions at the same place is equal to 0.

Two fermions cannot occupy the same state. This was the Pauli exclusion principle, which we needed to get the periodic table. Pauli. Two. If we have-- so this is fermions-- if we have bosons, ψ of a,b -- let me write this out.

So ψ fermion or boson-- so fermion is going to be with a minus and boson is going to be with a plus-- is equal to, suppose I have two particles $1/\sqrt{2}$. And one particle is in the state χ and the other particle is in the state ϕ . But in order to be fermionic or bosonic, in order for this to be invariant under swapping a and b , we have to have a plus or minus χ of b , ϕ of a .

And here we immediately see this Pauli principle at work. If I could take the fermionic example with the minus sign, then ψ evaluated at a, a must be χ at a , ϕ at b , minus χ at b , ϕ at a . But if a is b , this is χ a , ϕ a , minus χ a , ϕ a . That's zero.

But let's think about the bosonic case. If we have a bosonic field-- or, if we have-- sorry. If we have bosonic identical particles, then ψ b at a with a -- for our fermion, it was zero. But ψ b of the amplitude to be at two at the same place is equal to-- well, if b is a , then these two terms are identical and we have a plus. This is $\sqrt{2}$ χ at a , ϕ at a . Which is greater than what you might have naively guessed, which would have been just χ a , ϕ a .

For bosons, they really like being next to each other. They really like being in the same place. And this will eventually lead to lasers. So from this simple statistical property under swapping, picking up a minus sign, we get the Pauli principle, which gave us the periodic table. And is going to give us in the next lecture bands and solids in conductivity.

From the same principle but with a plus, and the persistence of this sine, from the persistence of the statistics, from the fact that we have two identical bosons, we get that they like to be in the same spot. And we'll get lasers and boson [INAUDIBLE] condensates.

And next time, we'll pick up with fermions in a periodic potential and we'll study solids and get to diamond.